

# *Skeptical review: Viscosity-Dependent Latent Space Structure in a PINN for Burger’s Equation: Analysis via PCA and Fractal Dimension with a Renormalization Group Analogy*

---

## Summary

The manuscript analyzes how a physical control parameter (viscosity,  $\nu$ ) is encoded in the internal representation of a Physics-Informed Neural Network (PINN) trained on Burgers’ equation. From a pretrained model, the authors extract a 10-dimensional hidden-layer activation (called a “latent space”; Sec. 2.1) over a fixed  $(x, t)$  grid ( $101 \times 103$ ) for 25 viscosity values, producing per- $\nu$  point clouds of 10,403 points in 10D. After per-viscosity standardization (Sec. 2.2, Sec. 3.1), the paper applies PCA (Secs. 2.3–2.4, Sec. 3.2) and Grassberger–Procaccia correlation-dimension estimation (Secs. 2.5, Sec. 3.3). Reported results suggest strong compression ( $\approx 3$  PCs for  $\geq 90\%$  variance and  $\approx 3\text{--}4$  PCs for  $\geq 95\%$  across  $\nu$ ), highly stable leading PC directions across neighboring viscosities (cosine similarities near 1; Sec. 3.2.2), a systematic reallocation of explained variance from PC1 toward PC2/PC3 as  $\nu$  increases, and correlation-dimension estimates around 1.5–1.75 with a non-monotonic peak at intermediate viscosities (Sec. 3.3). The authors interpret these trends via an RG-like analogy (Secs. 2.6, 3.4, 4). The topic (parameter-dependent representation geometry in PINNs) is timely and the analyses are interpretable, but the paper currently lacks essential specification of the PDE/setup and the PINN training, does not justify or test key methodological choices (notably per- $\nu$  standardization), and provides no uncertainty/robustness assessment for PCA and correlation-dimension estimates. The RG framing is intriguing but presently metaphorical and sometimes overstated relative to the empirical evidence.

## Strengths

- Clear and timely objective: characterize how a physical parameter (viscosity) is reflected in a PINN internal representation (Sec. 1, Sec. 2.1).
- Use of complementary, interpretable tools—PCA (linear variance structure) and correlation dimension (nonlinear geometric complexity)—applied systematically across 25  $\nu$  values (Secs. 2.3–2.5, Sec. 3.2–3.3).
- Empirical patterns are potentially interesting for interpretability: low effective dimension (3–4 PCs), strong apparent stability of leading directions, and systematic shift in explained-variance allocation with  $\nu$  (Sec. 3.2–3.2.2).
- The paper attempts to connect representation geometry to physics-level intuition (inviscid-like vs diffusion-dominated regimes) and proposes an RG-inspired viewpoint that could motivate future work (Secs. 3.4, 4).

- Overall workflow (reshape  $\rightarrow$  standardize  $\rightarrow$  PCA/cosine similarity  $\rightarrow$  correlation dimension) is easy to follow at a high level, and figures (Figs. 2–4) convey the intended trends clearly.

## Major issues

1. **The PDE/problem specification is not stated and the manuscript’s repeated claim of “2D Burgers” conflicts with the described data fields (x:101, t:103; no second spatial coordinate).** This prevents readers from validating what equation is solved, what “2D” means (2 spatial dims vs 1D-in-space + time), and what physical regimes are expected as  $\nu$  varies (Sec. 2.1, Sec. 3, Sec. 4). The dataset naming (e.g., “turbulence bundle”) further increases ambiguity about whether this is a single trajectory, a random-IC ensemble, forced/decaying Burgers, etc.

*Recommendation:* In Sec. 2.1, explicitly write the governing PDE(s) (including dimensionality, variables, and all terms), specify spatial/temporal domains, initial and boundary conditions, and clarify whether this is 1+1D Burgers (one space + time) or genuinely 2D-in-space Burgers (and if so, where the missing coordinate/fields are in the data). State whether each  $\nu$  corresponds to a single IC/trajectory or an ensemble, and reconcile any “turbulence” terminology with the actual setup.

2. **The PINN and training setup are under-specified, making it impossible to assess whether the latent-space trends reflect physics/learning or artifacts of a particular model/run.** Missing: architecture (depth/width/activations), loss terms and weights (PDE residual vs data/IC/BC), collocation and sampling strategy, optimizer schedule, stopping criteria, training diagnostics, and—crucially—whether a single multi- $\nu$  model was trained ( $\nu$  as an input) or separate models per  $\nu$  (Sec. 2.1, Sec. 4).

*Recommendation:* Expand Sec. 2.1 (or add a dedicated Methods subsection) to fully document the PINN: architecture (including where the 10D layer sits and whether it is pre/post activation), all loss components and weights, sampling/collocation details, optimizer and learning-rate schedule, training duration and convergence diagnostics, and whether training is joint across all  $\nu$  or separate per  $\nu$ . Provide at least basic solution-quality validation across viscosities (e.g., PDE residual statistics, BC/IC error, or comparison to a reference solver for a few  $\nu$ ) so the latent analysis is grounded in accurate solutions.

3. **The object of study (“10D latent space”) is not adequately defined or justified.** The manuscript does not clearly identify which layer is used, why that layer is representative, whether it is a bottleneck vs simply a hidden layer of width 10, and whether similar conclusions hold for other layers (Sec. 2.1, Sec. 3.2–3.3). This limits interpretability and generality.

*Recommendation:* In Sec. 2.1, precisely identify the layer (index/depth; pre- vs post-nonlinearity; activation function) and justify why its activations are treated as “latent.” Add a minimal layer-wise comparison (e.g., one earlier and one later layer) to test whether (i) low effective dimension, (ii) “stable” PC orientations, and (iii) correlation-dimension behavior persist. If not feasible, explicitly scope conclusions to this layer in Sec. 4.

4. **Per-viscosity standardization (Sec. 2.2) materially changes what PCA measures and may affect cross- $\nu$  comparisons.** Standardizing each  $\nu$  separately makes PCA reflect correlation structure rather than absolute variance/scales, and can alter explained-variance trends and PC alignment across viscosities. The paper does not discuss these implications or provide sensitivity checks (Sec. 2.2–2.4, Sec. 3.2).

*Recommendation:* Explicitly state, in Sec. 2.2–2.4 and again in Sec. 3.2, that PCA is performed on per- $\nu$  standardized activations and interpret EVR trends accordingly. Add a sensitivity analysis comparing at least: (i) per- $\nu$  standardization (current), (ii) global standardization using pooled mean/std across all  $\nu$ , and ideally (iii) no standardization (with careful interpretation). Consider adding a ‘pooled PCA’ (fit PCA on all  $\nu$  jointly) and then analyze how per- $\nu$  covariance/EVR projects onto this global basis; this directly tests the “stable basis” claim.

5. **PCA “stability” claims (cosine similarity of  $\text{PC}_k$  between  $\nu_i$  and  $\nu_j$ ) are potentially confounded by eigenvalue near-degeneracies and ordering/permutation ambiguity.** High cosine similarity can be misleading if PC2–PC4 eigenvalues are close and eigenvectors rotate within a near-degenerate subspace; comparing only successive viscosities can also mask cumulative drift (Sec. 2.4, Sec. 3.2.2, Table 3).

*Recommendation:* In Sec. 3.2.2, report eigenvalue gaps (e.g.,  $\lambda_k/\lambda_{k+1}$  or  $\lambda_k - \lambda_{k+1}$ ) to show when individual PCs are well-defined. Replace/augment per-component cosine similarities with subspace similarity metrics (principal angles) for the span of the top- $m$  PCs. Add an all-pairs similarity heatmap or similarity-to-a-fixed-reference- $\nu$  plot to detect long-range drift. Keep the sign-handling (absolute cosine) but also handle potential component swaps by matching PCs via maximum absolute dot product when eigenvalues are close.

6. **The Grassberger–Procaccia correlation-dimension estimation is insufficiently specified and lacks uncertainty/robustness analysis, yet it supports a central claim (non-monotonic  $D_2$  peak at intermediate  $\nu$ ).** Key missing details include: distance metric; whether standardized or raw latent points are used;  $\epsilon$  range and sampling; how the scaling region is selected; handling of strong spatial/temporal correlations in the  $(x, t)$  grid; and any computational approximations for  $N \approx 10^4$  (Secs. 2.5, 3.3).

*Recommendation:* Substantially expand Sec. 2.5 to document the exact implementation: metric,  $\epsilon$  grid (min/max, log spacing, number of radii), scaling-region selection procedure (e.g., sliding-window fits with  $R^2$  thresholds), and computational approach (full pairs vs subsampled pairs/k-d tree). Because points come from a structured  $(x, t)$  grid, incorporate correlation handling (e.g., Theiler window in time, spatial subsampling, or block bootstrap). In Sec. 3.3, add uncertainty quantification (bootstrap over points/blocks and over fit windows) and include representative  $\log C(\epsilon)$  vs  $\log \epsilon$  plots with the fitted scaling region for low/intermediate/high  $\nu$  (appendix acceptable). Also test sensitivity of  $D_2$  to  $\epsilon$ -range choices and to subsampling/resolution of the  $(x, t)$  grid.

7. **The reported intrinsic dimension ( $\approx 1.5$ – $1.75$ ) and PCA effective dimension ( $\approx 3$ – $4$ ) are discussed as if directly comparable, but the sampling geometry strongly constrains the latent point cloud:** for fixed  $\nu$ ,  $\text{latent}(x, t; \nu)$  is the image of a 2D parameter domain  $(x, t)$  under a smooth map, so intrinsic dimension is expected to be  $\leq 2$  in generic settings. Without addressing this “domain-mapping” viewpoint, the correlation-dimension results may be an artifact of structured sampling rather than evidence of an emergent manifold complexity trend with  $\nu$  (Sec. 3.3–3.4).

*Recommendation:* In Sec. 3.3–3.4, explicitly discuss that the point cloud is generated by a mapping from a 2D grid  $(x, t)$  and is not i.i.d.; explain how this naturally yields  $D_2$  near 2 (or below due to correlations/finite-size effects). To strengthen interpretation, test whether  $D_2$  is stable under changes in grid resolution and under random subsampling of  $(x, t)$  points. Clarify why PCA may require  $> 2$  components to capture variance (e.g., curved 2D surface embedded in 10D), and separate this geometric explanation from any stronger “RG-like” claims.

8. **The RG analogy is presented as a central interpretive lens but remains metaphorical; the manuscript does not define an explicit coarse-graining or a flow with RG-like properties (semigroup structure, fixed points, universality).** Current wording risks over-claiming relative to the presented evidence (Secs. 2.6, 3.4, 4).

*Recommendation:* Revise Sec. 3.4 and Sec. 4 to clearly label the RG connection as heuristic and specify the limited mapping being proposed (e.g.,  $\nu$  as a control/scale-like parameter; leading latent modes as “effective” degrees of freedom). Remove or soften language suggesting a rigorous RG correspondence unless you add an explicit coarse-graining/transformation and demonstrate RG-like behavior (e.g., fixed-point-like stabilization, monotone flow of a quantity) on the latent statistics.

9. **Reproducibility is currently too limited for a scientific contribution: missing code/data availability statements; incomplete description of the exact pipeline from the .npz bundle to figures/tables; and corrupted/placeholder artifacts undermine confidence in reported numbers (Secs. 2.3–2.5, Sec. 3.1–3.2.1, Sec. 4).**

*Recommendation:* Add a Data/Code Availability section (Sec. 4 or end matter) stating whether the trained model, activation bundles, and analysis scripts will be released. Provide a concise end-to-end pipeline description or pseudocode (loading, reshaping, standardization conventions, PCA implementation details, cosine-similarity computation,  $D_2$  estimation settings). Ensure all tables/figures are regenerated from source outputs and remove placeholders/corrupt entries before submission.

## Minor issues

1. Tables and cross-references contain OCR/editing artifacts (e.g., “??” references; corrupted numeric cells in Table 2; encoding glitches in cosine-similarity tables), which blocks verification of claims (Sec. 3.1–3.2.2).

*Recommendation:* Systematically replace all “??” with correct figure/table numbers and regenerate Table 2 (and any affected tables) from the original computation outputs so every EVR entry is numeric. Fix text-encoding issues (e.g., “0-ı1”) and move any prose embedded in table cells into the main text.

2. Figures 2–4 should better match the underlying data types and include clearer axis/unit conventions. In particular, integer-valued PC-count curves should not be drawn as smooth continuous lines; viscosity axis scaling/units are not always explicit; and captions omit key methodological context (Figs. 2–4).

*Recommendation:* In Fig. 2, use step plots or discrete markers (with optional jitter) for PC counts; label  $\nu$  with units/meaning and indicate whether  $\nu$  is linearly or log spaced; annotate key  $\nu$  regions discussed in text. In Figs. 2–4, ensure captions state sample size (10,403 points per  $\nu$ ), standardization choice, and whether lines are raw values vs smoothed/interpolated.

3. Uncertainty quantification is missing from the presentation: EVRs, cosine similarities, effective dimensionalities, and  $D_2$  values are reported as single deterministic curves/tables (Sec. 3.2–3.3; Figs. 2–4).

*Recommendation:* Add bootstrap/subsampling-based error bars or confidence bands to Figs. 2–4 and summarize variability in the text (Sec. 3.2–3.3). Even a limited robustness study on a few representative  $\nu$  values would materially improve credibility.

4. The choice and interpretation of “effective dimensionality” thresholds (90% and 95% cumulative variance) are only lightly motivated (Sec. 3.2.1).

*Recommendation:* Briefly justify the thresholds (e.g., common practice or reconstruction accuracy needs) and add a compact table listing, for each viscosity index, the minimum number of PCs to reach 90% and 95% cumulative variance (or place it in an appendix).

5. Physical interpretability remains largely speculative: the paper does not connect latent PCs or  $D_2$  trends to measurable properties of the Burgers solutions (shock width, gradient norms, dissipation, spectra), making regime-based interpretations difficult to assess (Sec. 3.3–3.4).

*Recommendation:* Add targeted physics diagnostics for a few  $\nu$  values and correlate them with PC scores and/or  $D_2$  (e.g., correlation between PC1 score and  $\max|\partial u/\partial x|$ , shock thickness proxies, or dissipation). If such analysis is out of scope, explicitly label physical-regime statements as hypotheses and tone down causal language.

6. The manuscript lacks a focused related-work discussion situating PCA/intrinsic-dimension analyses of neural representations and interpretability work in PINNs, and clarifying novelty relative to prior RG-inspired ML interpretations (Sec. 1).

*Recommendation:* Add a short related-work subsection (e.g., Sec. 1.1) covering: interpretability/representation analysis in PINNs; PCA/intrinsic-dimension methods in deep representations; and RG-inspired viewpoints in ML/physics-informed modeling. Explicitly state what is new here (e.g.,  $\nu$ -resolved latent analysis combining PCA + correlation dimension on a PINN).

## Very minor issues

1. Notation/terminology is inconsistent (e.g., “Burger’s” vs “Burgers’”;  $\nu$  vs  $v$ ; inconsistent capitalization of “correlation dimension”) (Secs. 1–4).

*Recommendation:* Standardize to a single convention throughout (typically “Burgers’ equation” and  $\nu$  for viscosity) and apply consistent capitalization.

2. Section/formatting issues reduce readability: inconsistent heading styles (e.g., stray Markdown markers), hyphenation/line-break artifacts splitting words, and lingering placeholder references (Secs. 2.5–2.6, 3.1–3.4, 4).

*Recommendation:* Proofread and clean formatting: unify heading styles, remove line-break word splits, and ensure all cross-references resolve to existing figures/tables.

3. The standardization formula does not specify whether mean/std are computed columnwise (per latent feature) and how broadcasting is applied (Sec. 2.2).

*Recommendation:* Clarify that mean and std are 10D vectors computed per latent dimension over the 10,403 samples for each  $\nu$  (or globally, if added), and that normalization is applied elementwise per feature.

4. Metadata/presentation details appear mismatched to content (e.g., potentially irrelevant keywords; dataset/file naming that implies “turbulence” without explanation).

*Recommendation:* Align keywords with the manuscript’s actual scope and add a brief note clarifying any dataset naming conventions if they could mislead readers about the physical setup.

## Mathematical consistency audit

This section audits **symbolic/analytic** mathematical consistency (algebra, derivations, dimensional/unit checks, definition consistency).

**Maths relevance:** light

The paper contains light but nontrivial mathematics centered on standard PCA definitions (standardization, covariance eigendecomposition, explained variance ratios, cumulative variance thresholds) plus cosine similarity for comparing eigenvectors across viscosities and the Grassberger–Procaccia correlation dimension scaling law. There are no long derivations, and the governing Burgers equation is not explicitly written, limiting audit of PDE-related consistency.

### Checked items

1. ✓ **Feature count and tensor shape consistency** (Sec. 2.1, p.2)

- **Claim:** The data bundle has shape  $(101, 103, 25, 13)$  with 13 features =  $(x, t, \text{viscosity}, 10 \text{ latent dims})$ .
- **Checks:** definition consistency, dimensional bookkeeping
- **Verdict:** PASS; confidence: high; impact: minor
- **Assumptions/inputs:** Feature axis includes exactly one  $x$  coordinate, one time coordinate, one viscosity value, and 10 latent activations.
- **Notes:**  $1 + 1 + 1 + 10 = 13$  matches the stated feature count, consistent with the provided tensor shape.

2. ✓ **Latent feature indexing yields 10 dimensions** (Sec. 2.1, p.2)

- **Claim:** Latent dimensions occupy feature indices 3 through 12, producing a  $(101, 103, 25, 10)$  tensor.
- **Checks:** index arithmetic, definition consistency
- **Verdict:** PASS; confidence: high; impact: minor
- **Assumptions/inputs:** Indexing is inclusive and 0-based as implied by 13 total features.
- **Notes:** Indices 3..12 inclusive contain 10 entries, consistent with extracting 10 latent dimensions.

3. ✓ **Per-viscosity reshape to point cloud size 10403** (Sec. 2.1, p.2)

- **Claim:** For viscosity index  $i$ ,  $(101, 103, 10)$  is reshaped into  $(101 \times 103, 10) = (10403, 10)$ .
- **Checks:** shape arithmetic
- **Verdict:** PASS; confidence: high; impact: minor

- **Assumptions/inputs:** Flattening over the  $(x, t)$  grid preserves latent dimension as columns.
- **Notes:**  $101 \times 103 = 10403$ , consistent with the stated sample count per viscosity.

#### 4. ✓ **Standardization formula and intent** (Sec. 2.3, p.3)

- **Claim:** Standardize each latent dimension per viscosity by subtracting column mean and dividing by column std:  $X^{\{\text{rm std}\}}_{\{\nu_i\}} = (X)_{\cdot} - \operatorname{mean}(X_{\{\nu_i\}}) / \operatorname{std}(X_{\{\nu_i\}})$
- **Checks:** algebra, definition consistency
- **Verdict:** PASS; confidence: high; impact: minor
- **Assumptions/inputs:**  $\operatorname{mean}(\cdot)$  and  $\operatorname{std}(\cdot)$  are computed column-wise across the 10403 samples, division is elementwise by feature
- **Notes:** The expression matches standard per-feature z-scoring; only minor clarity about broadcasting is missing.

#### 5. ✓ **PCA eigendecomposition outputs and EVR definition** (Sec. 2.3, p.3)

- **Claim:** PCA on standardized data computes covariance eigenvalues/eigenvectors;  $\operatorname{EVR}_k = \lambda_k / \sum_j \lambda_j$ .
- **Checks:** definition consistency, normalization sanity
- **Verdict:** PASS; confidence: high; impact: minor
- **Assumptions/inputs:** Covariance matrix is computed from standardized data (exact normalization  $1/n$  vs  $1/(n-1)$  not specified)
- **Notes:** EVR definition is correct and invariant to covariance normalization choice; eigenvectors are stated to be orthonormal, consistent with covariance eigendecomposition.

#### 6. ✓ **Cumulative variance thresholds vs Table 2** (Sec. 3.2.1 and Table 2, p.5)

- **Claim:** At low viscosities, 3 PCs reach 95% variance; above  $\nu \approx 0.12$ , 4 PCs are needed for 95%, while 3 PCs always suffice for 90%.
- **Checks:** internal arithmetic consistency (symbolic, from printed ratios)
- **Verdict:** PASS; confidence: medium; impact: moderate
- **Assumptions/inputs:** Table 2 EVRs are representative of the computed PCA outputs
- **Notes:** From Table 2,  $\nu = 0.0100$ :  $\operatorname{PC1} + \operatorname{PC2} + \operatorname{PC3} = 0.7095 + 0.1578 + 0.1059 = 0.9732 (> 0.95)$  so 3 PCs for 95% is consistent;  $\nu = 0.1212$ :  $\operatorname{sum} = 0.9496 (< 0.95)$  so needing 4 PCs is consistent. The global claim “3 PCs always for 90%” cannot be fully verified without the full per-viscosity table, but is not contradicted by the shown entries.

#### 7. ✓ **Cosine similarity formula and unit-vector simplification** (Sec. 2.4.2, p.3)

- **Claim:**  $\cos(\mathbf{a}, \mathbf{b}) = (\mathbf{a} \cdot \mathbf{b}) / (||\mathbf{a}|| ||\mathbf{b}||)$  simplifies to  $\mathbf{a} \cdot \mathbf{b}$  for unit principal components; absolute value accounts for sign ambiguity.
  - **Checks:** algebra, constraint consistency
  - **Verdict:** PASS; confidence: high; impact: minor
  - **Assumptions/inputs:** Principal components are normalized to unit length
  - **Notes:** The simplification is correct under unit-norm; the absolute value correctly addresses the  $\pm$  sign invariance of eigenvectors.
8.  $\Delta$  **PC stability metric may miss permutations** (Sec. 2.4.2 and Sec. 3.2.2, pp.3–6)
- **Claim:** Comparing the  $k$ -th PC at  $\nu_i$  to the  $k$ -th PC at  $\nu_{i+1}$  via absolute cosine similarity measures directional stability.
  - **Checks:** logical sufficiency of metric, assumption audit
  - **Verdict:** UNCERTAIN; confidence: medium; impact: moderate
  - **Assumptions/inputs:** PC ordering by eigenvalues is consistent across viscosities
  - **Notes:** The metric is mathematically correct given fixed component identity, but the paper does not justify that PC  $k$  corresponds to the same mode across viscosities when eigenvalues are close or cross; sign is handled but potential swaps are not.
9.  $\checkmark$  **Correlation sum scaling and correlation dimension estimate** (Sec. 2.5, p.4)
- **Claim:** For a fractal manifold of dimension  $D$ , correlation sum scales as  $C(\epsilon) \propto \epsilon^D$  for small  $\epsilon$ , so  $D$  is the slope of  $\log C(\epsilon)$  vs  $\log \epsilon$  in a scaling region.
  - **Checks:** definition consistency
  - **Verdict:** PASS; confidence: high; impact: minor
  - **Assumptions/inputs:** Distances are computed in the chosen feature space with a fixed metric, A valid scaling region exists
  - **Notes:** The stated scaling relation and slope-based estimation are internally consistent as a definition/estimation procedure.
10.  $\checkmark$  **Standardized vs raw data for fractal dimension** (Sec. 2.5, p.4 and Sec. 3.1 / Sec. 3.3, pp.4–6)
- **Claim:** Fractal dimension estimation is performed on standardized datasets for consistency of scale across dimensions.
  - **Checks:** definition consistency across sections
  - **Verdict:** PASS; confidence: medium; impact: minor
  - **Assumptions/inputs:** The same standardization as PCA is applied per viscosity

- **Notes:** Methods mention either raw or standardized as possible; Results and Sec. 3.1 explicitly state subsequent fractal-dimension analyses used standardized data, which is consistent though the Methods wording could be made explicit.
11. ✓ **Claim that fractal dimension can be below PCA effective dimensionality** (Sec. 3.3, p.6)
- **Claim:** Estimated correlation dimensions ( $\sim 1.5\text{--}1.75$ ) are lower than embedding dimension 10 and generally lower than PCA effective dimensionality (3–4), supporting a lower-dimensional non-linear manifold.
  - **Checks:** conceptual consistency
  - **Verdict:** PASS; confidence: medium; impact: minor
  - **Assumptions/inputs:** PCA effective dimensionality refers to number of PCs to reach a variance threshold, not a strict intrinsic dimension
  - **Notes:** No internal contradiction: a correlation (fractal) dimension may differ from and be lower than a linear variance-based PCA count; the statement is qualitative and consistent with the definitions provided.
12. △ **Use of “2D Burger’s equation” vs described coordinates** (Abstract and Sec. 2.1, pp.1–2)
- **Claim:** The analyzed PINN solves the 2D Burger’s equation, and the dataset includes the necessary coordinates/fields.
  - **Checks:** definition consistency
  - **Verdict:** UNCERTAIN; confidence: medium; impact: critical
  - **Assumptions/inputs:** “2D” means two spatial dimensions unless otherwise defined
  - **Notes:** The dataset description lists only  $x$  and  $t$  (plus viscosity and latent features) and does not define the Burgers PDE or a second spatial coordinate. Without an explicit PDE/variable definition, the dimensionality claim cannot be verified and appears inconsistent with the stated data fields.

### Limitations

- The governing Burgers equation is not written explicitly anywhere in the provided PDF text, so PDE-level mathematical consistency (terms, dimensionality, boundary/initial conditions, meaning of “2D”) cannot be audited.
- Many figure/table references are placeholders (“??”), limiting traceability of some analytic claims to specific numbered objects.
- Only the text as provided in the attached 8 pages was available; no appendices or supplementary derivations were present to verify omitted steps (e.g., details of covariance definition, correlation-sum normalization, scaling-region selection).

## Numerical results audit

This section audits **numerical/empirical** consistency: reported metrics, experimental design, baseline comparisons, statistical evidence, leakage risks, and reproducibility.

17 numeric checks were executed: 16 PASS and 1 UNCERTAIN. Verified items include dataset/latent tensor dimensional consistency, multiple PCA cumulative explained-variance threshold classifications from Table 2 (3 PCs for 90% throughout the checked rows; 95% switching from 3 to 4 PCs at higher viscosities in the checked rows), percent-to-fraction conversions for EVR values, a minimum-threshold check on the listed cosine similarities in Table 3, and selected-range/peak pattern checks for fractal dimension values from Table 4. One prose-vs-table example (Table 1) could not be verified from the provided inputs.

### Checked items

1. ✓ **C01\_shape\_feature\_product** (p.2, Sec. 2.1 (Data Description and Preparation))
  - **Claim:** Data bundle shape is  $(101, 103, 25, 13)$  corresponding to  $x = 101$  points,  $t = 103$  points, viscosity indices=25, features=13.
  - **Checks:** dimension\_consistency
  - **Verdict:** PASS
  - **Notes:** Exact axis-size equality check.
2. ✓ **C02\_latent\_slice\_indices\_count** (p.2, Sec. 2.1)
  - **Claim:** The 10-dimensional latent space occupies feature indices 3 through 12, yielding 10 components and tensor shape  $(101, 103, 25, 10)$ .
  - **Checks:** index\_range\_to\_count
  - **Verdict:** PASS
  - **Notes:** Inclusive index range count.
3. ✓ **C03\_samples\_per\_viscosity\_product** (p.2, Sec. 2.1 (end))
  - **Claim:** Total number of samples per viscosity is  $101 \times 103 = 10403$ .
  - **Checks:** arithmetic\_product
  - **Verdict:** PASS
  - **Notes:** Integer multiplication.
4. △ **C04\_eda\_stddev\_change\_example** (p.4, Sec. 3.1)
  - **Claim:** Example: at  $\nu = 0.01$ , L1 std dev is 0.980; at  $\nu = 1.0$ , L1 std dev is 0.629 (Table 1 illustrative).
  - **Checks:** table\_text\_consistency
  - **Verdict:** UNCERTAIN
  - **Notes:** Table 1 values to compare against are not provided in inputs; cannot verify prose-vs-table consistency.

5. ✓ **C05\_table2\_row\_sum\_evr\_nu001** (p.5, Table 2 (row Visc. Idx 0,  $\nu = 0.0100$ ))
- **Claim:** At  $\nu = 0.01$ , EVR PC1=0.7095, PC2=0.1578, PC3=0.1059; 3 PCs suffice for 90% and 95%.
  - **Checks:** cumulative\_variance\_threshold
  - **Verdict:** PASS
  - **Notes:** Computed cumulative EVR over first 3 PCs; inferred minimum PCs within first 3.
6. ✓ **C06\_table2\_row\_sum\_evr\_nu00215** (p.5, Table 2 (row Visc. Idx 4,  $\nu = 0.0215$ ))
- **Claim:** At  $\nu = 0.0215$ , EVR PC1=0.6980, PC2=0.1665, PC3=0.1084; 3 PCs suffice for 90% and 95%.
  - **Checks:** cumulative\_variance\_threshold
  - **Verdict:** PASS
  - **Notes:** Computed cumulative EVR over first 3 PCs; inferred minimum PCs within first 3.
7. ✓ **C07\_table2\_row\_sum\_evr\_nu00562** (p.5, Table 2 (row Visc. Idx 9,  $\nu = 0.0562$ ))
- **Claim:** At  $\nu = 0.0562$ , EVR PC1=0.6477, PC2=0.1918, PC3=0.1282; 3 PCs suffice for 90% and 95%.
  - **Checks:** cumulative\_variance\_threshold
  - **Verdict:** PASS
  - **Notes:** Computed cumulative EVR over first 3 PCs; inferred minimum PCs within first 3.
8. ✓ **C08\_table2\_row\_sum\_evr\_nu01** (p.5, Table 2 (row Visc. Idx 12,  $\nu = 0.1000$ ))
- **Claim:** At  $\nu = 0.1$ , EVR PC1=0.5500, PC2=0.2438, PC3=0.1612; 3 PCs suffice for 90% and 95%.
  - **Checks:** cumulative\_variance\_threshold
  - **Verdict:** PASS
  - **Notes:** Computed cumulative EVR over first 3 PCs; inferred minimum PCs within first 3.
9. ✓ **C09\_table2\_row\_sum\_evr\_nu01212\_threshold\_shift** (p.5, Table 2 (row Visc. Idx 13,  $\nu = 0.1212$ ) and p.5 prose about shift starting index 13)
- **Claim:** At  $\nu \approx 0.1212$  (index 13), 3 PCs for 90% but 4 PCs for 95%; EVR PC1=0.5142, PC2=0.2611, PC3=0.1743.
  - **Checks:** cumulative\_variance\_threshold

- **Verdict:** PASS
  - **Notes:** Computed  $c3 = 0.9496$  ( $< 0.95$ ), consistent with requiring at least 4 PCs for 95% while still meeting 90% with 3 PCs.
10. ✓ **C10\_table2\_row\_sum\_evr\_nu03162** (p.5, Table 2 (row Visc. Idx 18,  $\nu = 0.3162$ ))
- **Claim:** At  $\nu = 0.3162$ , 3 PCs for 90% and 4 PCs for 95%; EVR PC1=0.4422, PC2=0.2573, PC3=0.2288.
  - **Checks:** cumulative\_variance\_threshold
  - **Verdict:** PASS
  - **Notes:** Computed  $c3 = 0.9283$  ( $< 0.95$ ), consistent with 4 PCs needed for 95% while 3 PCs suffice for 90%.
11. ✓ **C11\_table2\_row\_sum\_evr\_nu1** (p.5, Table 2 (row Visc. Idx 24,  $\nu = 1.0000$ ) and p.5 prose)
- **Claim:** At  $\nu = 1.0$ , EVR PC1=0.3929, PC2=0.3249, PC3=0.2175; 3 PCs for 90% and 4 PCs for 95%.
  - **Checks:** cumulative\_variance\_threshold
  - **Verdict:** PASS
  - **Notes:** Computed  $c3 = 0.9353$  ( $< 0.95$ ), consistent with 4 PCs needed for 95% while 3 PCs suffice for 90%.
12. ✓ **C12\_percent\_to\_fraction\_consistency\_pc1** (p.5, Sec. 3.2.1 prose + Table 2 ( $\nu = 0.01, 0.1, 1.0$ ))
- **Claim:** PC1 accounts for approximately 70.95% at  $\nu = 0.01$ , 55.00% at  $\nu = 0.1$ , 39.29% at  $\nu = 1.0$ ; Table 2 lists 0.7095, 0.5500, 0.3929.
  - **Checks:** percent\_fraction\_conversion
  - **Verdict:** PASS
  - **Notes:** All percent/100 conversions match fractions within tolerance.
13. ✓ **C13\_percent\_to\_fraction\_consistency\_pc2\_pc3** (p.5, Sec. 3.2.1 prose + Table 2 ( $\nu = 0.01, 0.1, 1.0$ ))
- **Claim:** PC2 rises 15.78%  $\rightarrow$  24.38%  $\rightarrow$  32.49% and PC3 rises 10.59%  $\rightarrow$  16.12%  $\rightarrow$  21.75%; Table 2 lists 0.1578/0.2438/0.3249 and 0.1059/0.1612/0.2175.
  - **Checks:** percent\_fraction\_conversion
  - **Verdict:** PASS
  - **Notes:** All percent/100 conversions match fractions within tolerance.
14. ✓ **C14\_pc\_stability\_claim\_vs\_table3\_min** (p.5-6, Sec. 3.2.2 prose + Table 3)
- **Claim:** Cosine similarities for PC1, PC2, PC3 are generally exceeding 0.988; Table 3 lists selected Sim(PC1/2) values including 0.9898 and 0.9885.

- **Checks:** threshold\_check\_on\_listed\_values
  - **Verdict:** PASS
  - **Notes:** Checked minimum of listed cosine similarities against threshold (min listed = 0.9885).
15. ✓ **C15\_table3\_pairwise\_symmetry\_pc1\_pc2** (p.6, Table 3)
- **Claim:** For viscosity pairs 13-14 and 14-15,  $\text{Sim}(\text{PC1}) = \text{Sim}(\text{PC2})$  as printed (0.9898 and 0.9885 respectively).
  - **Checks:** equality\_within\_rounding
  - **Verdict:** PASS
  - **Notes:** Equality check for printed  $\text{Sim}(\text{PC1})$  vs  $\text{Sim}(\text{PC2})$  pairs.
16. ✓ **C16\_table4\_range\_claim** (p.6, Sec. 3.3 prose + Table 4)
- **Claim:** Fractal dimension ranges approximately 1.50 to 1.75 across all viscosities; Table 4 selected values include 1.4986 (min shown) and 1.7499 (max shown).
  - **Checks:** min\_max\_range\_on\_listed\_values
  - **Verdict:** PASS
  - **Notes:** Checked min/max of listed FD values against stated approximate range window.
17. ✓ **C17\_table4\_peak\_nonmonotonicity\_selected** (p.6, Sec. 3.3 prose + Table 4)
- **Claim:** Non-monotonic fractal dimension: increases from 1.4986 (idx0) up to peak around 1.75 (idx13-14), then decreases to 1.5325 (idx24).
  - **Checks:** nonmonotonic\_pattern\_on\_listed\_values
  - **Verdict:** PASS
  - **Notes:** Strict inequality peak check on selected values.
18. ✓ **C18\_effective\_dimensionality\_vs\_embedding\_dim** (p.5, Sec. 3.2.1 + p.6, Sec. 3.3)
- **Claim:** Embedding dimension is 10; PCA effective dimensionality is 3-4; fractal dimension 1.5-1.75 is lower than both.
  - **Checks:** inequality\_chain
  - **Verdict:** PASS
  - **Notes:** Checked claimed numeric ordering among dimensions.

## Limitations

- Checks are limited to arithmetic/logical consistency using numbers explicitly present in the provided PDF text/tables; the underlying NumPy dataset and full per-viscosity metric arrays are not available here.

- Figure captions describe trends, but numeric extraction from plotted curves is out of scope (no plot-pixel reading). Only tabulated/explicitly written numbers were used.
- Table 1 is labeled 'Illustrative' and is incomplete (ellipsis rows), limiting robust cross-checking beyond the explicitly shown entries.
- One executed check (C04) was UNCERTAIN because the Table 1 numeric entries needed to compare against the prose example were not available in the provided inputs.