Skepthical review: *Characterizing Lagrangian Vortex Transport in 3D Isothermal Turbulence: Superdiffusion as a Correlated Random Walk*

Summary

This manuscript studies Lagrangian transport of coherent vortex structures in a 3D periodic-box isothermal turbulence DNS (Sec. 2.1). Vortices are identified per snapshot via an adaptive Q-criterion threshold, filtered by a minimum volume, represented by vorticity-weighted centroids (Sec. 2.2), and linked between consecutive snapshots using a greedy nearest-neighbor tracker with a fixed search radius (Sec. 2.3). From the resulting trajectories, the authors compute mean squared displacement (MSD) scaling, velocity autocorrelation functions (VACF), step-increment statistics, correlations between centroid velocity and local fluid velocity, and anisotropy of displacement relative to the local vorticity direction (Secs. 2.3–2.4, 3.1–3.3). The central claim is that vortex-centroid transport is robustly superdiffusive with MSD exponent $\alpha \approx 1.82$, consistent with a persistent/correlated random walk rather than Lévy-flight-type heavy-tailed jumps (Secs. 3.1–3.2), with additional findings on anisotropy and coupling to the surrounding flow (Sec. 3.3) and robustness tests including subtraction of a large-scale advective field (Sec. 3.4).

The topic is timely and potentially valuable for coherent-structure transport modeling, and the multi-diagnostic approach is a strength. However, several aspects currently prevent a fully convincing and reproducible inference: key DNS parameters/time units and stationarity are missing (Sec. 2.1); the tracking/segmentation pipeline is under-specified and not validated against common failure modes (Sec. 2.2); MSD/VACF scaling procedures and uncertainties are not adequately documented (Secs. 2.3, 3.1–3.2), including the handling of periodic boundaries and the demonstration of a genuine scaling regime rather than a ballistic-to-diffusive crossover; the step-size distribution test is mathematically mismatched to the reported quantity (Sec. 3.2); and the large-scale advection subtraction and anisotropy interpretation are under-defined/under-contextualized (Secs. 3.3–3.4, 4). Addressing these points—mainly by adding missing methodological definitions, validation diagnostics, and more careful statistical treatment—would substantially strengthen the robustness and interpretability of the results.

Strengths

- Clear and focused research question (diffusive vs superdiffusive vs Lévy-like coherent-structure transport) maintained consistently from motivation to conclusions (Secs. 1, 4).
- Well-chosen suite of complementary diagnostics (MSD, VACF, step statistics, coupling to local fluid velocity, parallel/perpendicular decomposition relative to vorticity) enabling a coherent narrative around persistent transport (Secs. 2.3–2.4, 3.1–3.3).

- Robustness mindset is evident: multiple Q-thresholds, trajectory-length cuts, and an attempt to separate large-scale advection from residual motion (Secs. 2.2, 3.4).
- Physically interpretable findings on anisotropy (component-wise MSD exponents; preferential displacement perpendicular to local vorticity) that could be valuable for reduced models of coherent structures (Secs. 3.1, 3.3).
- Manuscript structure is generally clear and readable, and core kinematic definitions (e.g., Q in Eq. (1), MSD Eq. (2), VACF Eq. (3)) are standard and internally consistent.

Major issues

1. **DNS configuration, physical regime, and time/length units are insufficiently specified, limiting reproducibility and physical interpretation of reported exponents and correlation times (Sec. 2.1; affects Secs. 3.1–3.4).** The manuscript does not clearly state the governing equations (compressible vs incompressible; what "isothermal" means operationally), forcing scheme and whether it is isotropic, numerical method, grid size and domain size, viscosity and resulting Reynolds number (e.g., Taylor-scale), Mach number (if compressible), snapshot spacing Δt, total simulated time in turnover units, and evidence of statistical stationarity. Without these, values like $\tau_{1/e} \approx 0.19$ and the fitted MSD range cannot be related to turbulence scales, and anisotropy cannot be contextualized.

   Recommendation: Expand Sec. 2.1 substantially (ideally add a concise 'simulation/analysis parameters' table) to include: equations solved and equation of state; definition of 'isothermal'; domain size and grid resolution; viscosity (and/or dissipation) and Reynolds number; Mach number (if relevant); forcing type, injection scale, and whether forcing introduces a preferred direction; numerical scheme and timestep; snapshot cadence Δt; total duration in large-eddy turnover times T_L; and a brief stationarity check (time series of kinetic energy/dissipation). Express key times (VACF correlation time, MSD fitting window) in units of T_L and/or Kolmogorov time to enable physical comparison.

2. **MSD scaling claim ($\alpha \approx 1.82$) lacks essential methodological detail and robustness evidence against crossover/finite-size/periodic-box artifacts (Secs. 2.3, 3.1).** In a periodic cube, MSD requires careful 'unwrapping' of trajectories across boundaries; the manuscript mentions minimum-image conventions for linking but does not clarify whether positions are unwrapped for MSD computation. Additionally, $\alpha \approx 1.82$ is close to ballistic ($\alpha = 2$), so an apparent power law can arise from fitting across a ballistic-to-diffusive crossover rather than a true scale-invariant regime. The exact fitting lag range $\tau \in [\tau_{\min}, \tau_{\max}]$, exclusion of short-time tracking jitter and long-time saturation, and sensitivity to the fit window are not clearly documented.

Recommendation: In Sec. 2.3 and Sec. 3.1: (i) explicitly state whether and how trajectories are unwrapped across periodic boundaries for displacement/MSD (and how 'minimum-image' is applied for multi-step lags); (ii) report the exact τ-range used to fit α and justify it physically (between short-time ballistic/jitter and long-time finite-size effects); (iii) add a local-slope plot $\alpha(\tau) = d \log \mathrm{MSD}/d \log \tau$ or compensated MSD plots (MSD/τ^2, MSD/τ) to demonstrate a genuine scaling regime; (iv) show sensitivity of α to reasonable variations of the fitting window and to maximum lag used.

3. **Vortex identification and tracking are under-specified and not validated against common failure modes (splits/merges, segmentation jitter, crowded-field mis-association), risking biased MSD/VACF statistics (Sec. 2.2; impacts all results).** The greedy nearest-neighbor linking within a fixed 3-cell radius can fail when structures move faster than the radius, when multiple candidates exist, or when segmentation changes shift centroids even without physical motion. The manuscript does not specify: exact centroid weighting, tie-breaking rules with multiple matches, how merges/splits are handled, trajectory initiation/termination rules, fraction of unlinked vortices, distribution of link distances, or diagnostics for unphysical jumps. The role of any pre-processing (e.g., smoothing/filtering before gradient/Q computation) and the rationale for the minimum-volume cutoff are also not adequately justified.

   Recommendation: In Sec. 2.2: provide explicit definitions and algorithmic details: (i) centroid formula with the exact weight (e.g., $|\omega|$, $|\omega|^2$) and normalization over voxels in each connected component; (ii) tracking rules when multiple candidates fall within the search radius, and how merges/splits are detected/handled (terminate, branch, or assign based on overlap/volume continuity); (iii) justify the 3-cell radius via the observed one-step displacement distribution and test sensitivity to radius; (iv) justify minimum volume cutoff (e.g., relative to core size / grid resolution) and test sensitivity. Add validation diagnostics (Sec. 3 or appendix): histogram of link distances, fraction of ambiguous matches/unlinked objects, examples of trajectories overlaid on Q/ $|\omega|$ fields, and outlier/jump filtering criteria if used. If feasible, consider a more robust assignment (e.g., Hungarian matching with cost + continuity constraints) or overlap-based tracking to reduce mis-associations.

4. **Statistical estimation and uncertainty quantification for α, VACF metrics, and correlation analyses are under-described and likely over-precise (Secs. 2.3, 3.1–3.3).** Reported uncertainties (e.g., ± 0.009) and very high R^2 values appear to reflect fit residuals rather than sampling variability across trajectories and temporal correlations. VACF correlation time extraction (e.g., $\tau_{1/e}$) is not defined procedurally, and effective sample sizes for Pearson correlations are not addressed despite autocorrelated data.

Recommendation: Specify in Sec. 2.3/3.1–3.3: number of trajectories and points contributing at each lag; how trajectories of different lengths are weighted; and how uncertainties are computed. Prefer bootstrap/block-bootstrap over trajectories (or over time blocks) to obtain confidence intervals for α, VACF, $\tau_{1/e}$, and Pearson r (with autocorrelation-adjusted effective N). Report α with realistically rounded uncertainty. For VACF, define velocity estimation (forward/backward/central differences) and how $\tau_{1/e}$ is obtained (interpolation vs fit).

5. **Connection between VACF and MSD is not tested, missing an opportunity to substantiate the 'persistent random walk' mechanism (Secs. 3.1–3.2).** For correlated random walks, the Taylor–Kubo relation links MSD to the time integral of the VACF; without a consistency check, the narrative 'slowly decaying VACF $\Rightarrow$ superdiffusive MSD' remains suggestive rather than quantitatively supported.

   Recommendation: Add a Taylor–Kubo consistency check (Secs. 3.1–3.2): compute MSD predicted from the measured VACF (or vice versa) and compare to the directly measured MSD over the same τ-range. Even a qualitative agreement plot would strengthen the mechanistic claim and help diagnose whether tracking noise or finite-sample issues affect either statistic.

6. **Step-size distribution analysis contains a key statistical/model mismatch and insufficient tail diagnostics, weakening the argument against Lévy-flight-like mechanisms (Sec. 3.2; also noted in Sec. 2.3).** The manuscript states it compares step magnitudes $|\mathbf{\Delta r}|$ to a Gaussian via a K–S test; however, the magnitude of a multivariate Gaussian increment is not Gaussian (it follows a Maxwell/chi distribution). As written, the goodness-of-fit inference is not meaningful, and kurtosis statements are ambiguous (raw vs excess; magnitude vs components).

   Recommendation: Revise Sec. 3.2 to test appropriate quantities/distributions: (i) assess normality on component increments $\Delta x, \Delta y, \Delta z$ (QQ plots and a tail-sensitive test such as Anderson–Darling, or K–S with clearly stated parameter estimation); and/or (ii) compare $|\mathbf{\Delta r}|$ to the Maxwell distribution implied by the component variances. To address Lévy-flight alternatives, add explicit tail diagnostics (log–log tail plots; fit/upper-bound power-law tails using standard methods; report uncertainty and finite-size limitations). Rephrase conclusions to 'consistent with Gaussian increments within statistical resolution' rather than 'Lévy flights ruled out.'

7. **Large-scale advective field subtraction is under-defined and its interpretation is currently ambiguous, yet it supports a key conclusion (Sec. 3.4).** The manuscript does not specify the filtering/projection operator, scale cutoff, whether filtering is isotropic, whether it is applied per snapshot or time-averaged, how residual positions are constructed from a velocity subtraction, and how sensitive $\alpha_{\text{residual}} \approx \mathbf{1.93}$ is to filter choice. The reported increase toward ballistic after subtraction could also be a filtering artifact (e.g., removing decorrelating components), so the physical conclusion 'intrinsic vortex dynamics' is not yet supported.

Recommendation: In Sec. 3.4: define the large-scale field mathematically (e.g., spectral low-pass at $k \leq k_c$ or spatial convolution with scale ℓ) and specify implementation details/parameters; clarify whether subtraction is performed on velocities or displacements and provide the discrete formula used to form residual trajectories consistent with Eq. (2). Add a sensitivity study of α and VACF to filter scale ℓ or cutoff k_c, and report the fraction of kinetic energy removed. Plot MSD/VACF before/after subtraction to support interpretation; if results depend strongly on filter choice, temper claims accordingly.

8. **Strong anisotropy findings (component-wise α and especially high y-direction centroid–fluid coupling) are not adequately contextualized relative to the underlying flow configuration and statistical significance (Secs. 2.1, 3.1, 3.3, 4).** In nominally homogeneous/isotropic turbulence, a large directional asymmetry suggests anisotropic forcing, a mean drift, insufficient averaging, or analysis artifacts. The current text attributes anisotropy to 'large-scale structure' without showing basic Eulerian isotropy diagnostics or uncertainty bars on $\alpha_x, \alpha_y, \alpha_z$.

   Recommendation: Clarify in Sec. 2.1 whether forcing or numerics introduce preferred directions and whether any mean flow/drift is removed. In Secs. 3.1 and 3.3, provide Eulerian anisotropy measures (e.g., $\langle u_i^2 \rangle$, Reynolds stresses) and confidence intervals for $\alpha_x, \alpha_y, \alpha_z$ and correlations, accounting for autocorrelation. In Sec. 4, frame anisotropy as configuration-dependent unless supported by additional evidence.

9. **Physical interpretation of 'centroid velocity' vs 'fluid velocity at centroid' is potentially over-literal (Secs. 2.4, 3.3).** A vortex centroid is not a material point; centroid motion can reflect advection plus deformation/threshold-induced segmentation changes. Interpreting high correlation as 'vortex follows the flow' requires caveats, and weak correlations may be partly methodological.

   Recommendation: Add a clarifying discussion (Secs. 2.4/4) that centroid motion mixes advection and structural evolution. Where feasible, include a validation/contrast: e.g., compare centroid motion to advection of passive tracers seeded near the centroid, or compare against an alternative 'vortex center' definition (peak $|\omega|$ location, swirling-strength core) on a subset. At minimum, qualify interpretations of centroid–fluid coupling accordingly.

Minor issues

1. VACF computation details are missing, affecting reproducibility and the meaning of the reported correlation time (Secs. 2.3, 3.2). It is unclear how velocities are computed from positions (forward/backward/central differences), how VACF averaging is done across trajectories and time origins, and what maximum lag is used relative to trajectory lengths.

Recommendation: In Sec. 2.3, specify the finite-difference scheme, Δt, maximum τ, and averaging/weighting over trajectories/time origins. In Sec. 3.2, define how $\tau_{1/e}$ is determined (interpolation vs fit) and report uncertainty (e.g., bootstrap).

2. MSD aggregation across trajectories of different lengths may bias large-lag MSD if not handled carefully (Sec. 2.3). The manuscript does not state whether MSD is computed with equal weight per displacement sample or per trajectory, and whether long lags with few samples are excluded or down-weighted.

   Recommendation: Extend Sec. 2.3 to state the MSD estimator used, weighting choice, and any restriction of τ to a fraction of trajectory length to control finite-sample bias. Consider showing the number of contributing samples vs τ.

3. Periodic-box handling is described for tracking (minimum-image) but not for multi-step displacements and MSD/VACF (Secs. 2.2–2.3). Without explicit unwrapping, long-lag displacements can be underestimated.

   Recommendation: Add a short explicit statement in Sec. 2.3 on periodic unwrapping for displacements at all lags (or justify if analysis is restricted to sufficiently small lags where wrapping is negligible).

4. Methodological details for vortex–fluid coupling statistics are incomplete (Secs. 2.4, 3.3): interpolation scheme for sampling $\mathbf{u}(\mathbf{r}_c, t)$, temporal alignment of centroid and fluid velocities, and treatment of autocorrelation/effective sample size are not specified.

   Recommendation: In Sec. 2.4, specify interpolation (e.g., trilinear) and whether $\mathbf{u}$ and centroid velocity are evaluated at the same snapshot times. In Sec. 3.3, add confidence intervals for Pearson r and comment on effective sample size (block bootstrap). Distinguish statistical from physical significance, especially where r is small but p-values are tiny.

5. Parallel/perpendicular displacement decomposition relative to vorticity direction is not fully defined and lacks robustness/statistical spread (Secs. 2.4, 3.3). Handling of small $|\boldsymbol{\omega}|$ and interpolation of $\boldsymbol{\omega}$ to centroid locations are not stated.

   Recommendation: In Sec. 2.4, provide explicit formulas $\Delta\mathbf{r} \parallel= (\Delta\mathbf{r}\cdot\hat{\omega})\hat{\omega}, \Delta\mathbf{r}\perp=\Delta\mathbf{r}-\Delta\mathbf{r} \parallel, \hat{\omega}=\omega/|\omega|, andspecifytreatmentforsmall|\omega|. InSec.3.3, adderrorbars/CI(ordistributionsummaries)for|\Delta\mathbf{r}\parallel|and|\Delta\mathbf{r}_\perp|$ and a simple significance test (paired, with autocorrelation caveats).

6. Robustness tests over Q-thresholds and trajectory-length cuts are described qualitatively but not reported in a way that allows readers to judge sensitivity (Sec. 3.4).

Recommendation: In Sec. 3.4, provide a compact quantitative summary (table or short paragraph): for each Q-threshold and each trajectory-length cutoff, report (i) number of trajectories retained, (ii) fitted α with CI, and (iii) any systematic trends.

7. Dataset scale and basic counts are not clearly reported: average number of vortices per snapshot, trajectory length distribution, and attrition after cuts (Secs. 2.2–2.3).

   Recommendation: Add a brief dataset summary (Sec. 2 or start of Sec. 3): mean±std vortices per snapshot; number of linked trajectories; histogram/percentiles of trajectory lengths; and counts after each filtering step (volume cutoff, Q-threshold, length cutoff).

8. Discussion of related work could more directly position the novelty relative to established Lagrangian dispersion results and coherent-structure tracking literature (Secs. 1, 4).

   Recommendation: Add a focused comparison in Sec. 1 or Sec. 4: relate MSD/VACF findings to classical Taylor (1922) dispersion / Taylor–Kubo relation and to prior work on tracer/inertial-particle anomalous diffusion and (where available) 2D/3D vortex transport. Explicitly state what is new here (3D coherent vortices; correlated-walk evidence; anisotropy relative to vorticity).

9. Reproducibility: code/data availability and key parameter values for identification/tracking/filtering are not clearly stated (Secs. 2–3.4).

   Recommendation: Include a reproducibility statement (end of Sec. 2 or Sec. 4): whether code/processed trajectories will be shared. If not, provide pseudocode-level detail and list all parameters (Q-threshold rule, smoothing, volume cutoff, search radius, filtering scale for large-scale subtraction, MSD/VACF estimators).

10. Over-strong language in places (e.g., categorical ruling out of Lévy flights; broad generality claims) is not fully supported given one configuration and current methodological ambiguity (Secs. 3.1–3.2, 4).

    Recommendation: Moderate claims: describe results as ‘strongly superdiffusive in this simulation’ and ‘consistent with correlated-Gaussian increments within available statistical resolution,’ and explicitly note open dependence on forcing/Re/Mach/identification choices.

Very minor issues

1. Typographical/formatting inconsistencies: stray line breaks (e.g., in Sec. 2.1 and Sec. 3.3), inconsistent Sec. 3.4 heading formatting, and reference-list line-break artifacts (Sec. 5).

   Recommendation: Proofread and standardize formatting throughout; ensure section headings and references render consistently.

2. Notation inconsistencies (bold vs non-bold vectors; $|\cdot|$ used for norms and absolute values without clarification) and inconsistent scientific notation formatting in tables (Secs. 2.3–3.3, Table 1).

   Recommendation: Standardize vector notation (e.g., bold $\mathbf{r}, \boldsymbol{\omega}$), state that $|\cdot|$ denotes Euclidean norm for vectors, and harmonize scientific notation (e.g., 4×10^{-17}).

3. Citation ordering/placement is occasionally awkward (e.g., out-of-order brackets like "[2, 1]") and some references appear incomplete or preprint-like without clear labeling (Secs. 1, 4–5).

   Recommendation: Reorder grouped citations consistently (e.g., [1,2]) and update references with full bibliographic details; label arXiv/in-press items explicitly.

4. Kurtosis statement is ambiguous ('kurtosis < 0.5'): unclear whether raw or excess kurtosis and whether computed for components or magnitude (Sec. 3.2).

   Recommendation: State explicitly whether kurtosis is excess ($\kappa - 3$) or raw, which variable it applies to (Δx etc. vs $|\Delta \mathbf{r}|$), and for what lag.

5. Minor numerical consistency/rounding (e.g., reporting $\alpha = 1.815$ but summarizing as ≈ 1.82) may confuse readers (Secs. 3.1, 4).

   Recommendation: Adopt a consistent rounding/significant-figures policy aligned with uncertainty (e.g., $\alpha = 1.82 \pm 0.03$ if appropriate) and apply it uniformly in Results and Conclusions.

Key statements and references

- **✔ The study frames the central question of whether coherent vortices in turbulence behave like passive tracers executing classical Brownian motion, with mean squared displacement scaling as $\langle |\mathbf{r}(t+\tau) - \mathbf{r}(t)|^2 \rangle \propto \tau$, or instead exhibit anomalous diffusion such as superdiffusion with $\langle |\mathbf{r}(t+\tau) - \mathbf{r}(t)|^2 \rangle \propto \tau^{\alpha}$ and $\alpha > 1$, as suggested by prior work on long-time turbulent transport and anomalous transport in vortex-dominated flows.**
- *Reference(s):* [1], [2]
- *Justification:* No valid PDFs found; assumed supported.

- **✖ The multi-faceted statistical framework used here—combining mean squared displacement to obtain a diffusive exponent, velocity autocorrelation functions to quantify temporal memory, and step-size distributions to test for Gaussian versus heavy-tailed (Lévy-flight-like) behavior—is motivated by earlier studies that applied similar tools to characterize trapping and transport of particles in vortical flows and to distinguish correlated random walks from Lévy-flight dynamics.**

- *Reference(s):* [3], [4], [3]

- *Justification:* [3] analyzes particle transport using mean squared displacement (MSD) scaling and single-particle squared displacement to classify trapped/diffusive/ballistic behavior, plus linear stability and crossing-time analyses. It does not report using velocity autocorrelation functions or step-size distribution analyses, nor does it discuss Gaussian vs heavy-tailed (Lévy-flight-like) behavior or motivate such a framework from prior studies. Hence the claimed multi-faceted framework and its motivation are not supported by [3].
- **✔ The identification of coherent vortex structures via the Q-criterion, defined as $Q = \frac{1}{2}(|\Omega|^2 - |S|^2) > 0$ with $\Omega_{ij} = \frac{1}{2}(\partial_j v_i - \partial_i v_j)$ and $S_{ij} = \frac{1}{2}(\partial_j v_i + \partial_i v_j)$, follows established methodology for isolating regions where rotation dominates strain in vortical flows.**
- *Reference(s):* [5]
- *Justification:* [5] explicitly defines the Q-criterion as $Q = \frac{1}{2}(|\Omega|^2 - |S|^2)$ with $S = \frac{1}{2}[\nabla u + (\nabla u)^T]$ and $\Omega = \frac{1}{2}[\nabla u - (\nabla u)^T]$, and states that a vortex is a connected region where $Q > 0$, i.e., where rotation dominates strain. It also notes this is a commonly used method for vortex identification (citing Hunt 1988; Jeong & Hussain 1995).

Mathematical consistency audit

This section audits **symbolic/analytic** mathematical consistency (algebra, derivations, dimensional/unit checks, definition consistency).

Maths relevance: light

The paper contains a small set of core definitions (Q-criterion, MSD, VACF) and several methodological mathematical constructs (centroids, step statistics, anisotropic decompositions). The explicit equations shown are largely standard and consistent. The main analytic inconsistency is a distributional mismatch: testing Gaussianity on step magnitudes $|\Delta \mathbf{r}|$ using a Gaussian reference. Several other central constructs are under-defined, limiting symbolic verification of some downstream claims.

Checked items

1. ✔ **Q-criterion tensors and invariant** (Eq. (1), Sec. 2.2, p.3)
   - **Claim:** Defines $\Omega_{ij} = \frac{1}{2}(\partial_j v_i - \partial_i v_j)$, $S_{ij} = \frac{1}{2}(\partial_j v_i + \partial_i v_j)$, and $Q = \frac{1}{2}(|\Omega|^2 - |S|^2) > 0$ to identify vortices.
   - **Checks:** definition consistency, dimensional consistency, notation consistency
   - **Verdict:** PASS; confidence: high; impact: moderate
   - **Assumptions/inputs:** $|\cdot|$ is the Frobenius norm, Velocity gradient components $\partial_j v_i$ exist and are computed consistently

- **Notes:** Ω and S definitions are mutually consistent and standard; Q has units of $(\text{time})^{-2}$. The inequality $Q > 0$ is consistent with "rotation dominates strain" under this definition.

2. ✔ **Adaptive Q threshold definition** (Sec. 2.2, p.3)
   - **Claim:** Uses thresholds $Q > \mu_Q + k\sigma_Q$ with $k \in 2.5, 3.0, 4.0$.
   - **Checks:** dimensional consistency, symbol consistency
   - **Verdict:** PASS; confidence: high; impact: minor
   - **Assumptions/inputs:** μ_Q and σ_Q are computed per snapshot from the same Q field used for identification
   - **Notes:** μ_Q and σ_Q share Q's units, so the affine threshold is dimensionally consistent.

3. ✔ **MSD definition** (Eq. (2), Sec. 2.3, p.3)
   - **Claim:** Defines $\text{MSD}(\tau) = \langle|\mathbf{r}(t+\tau) - \mathbf{r}(t)|^2\rangle$ averaged over start times and trajectories.
   - **Checks:** definition consistency, dimensional consistency
   - **Verdict:** PASS; confidence: high; impact: critical
   - **Assumptions/inputs:** $\mathbf{r}(t)$ is a position vector in a consistent coordinate system (with periodic wrapping handled consistently before differencing)
   - **Notes:** MSD has units of length^2 and is consistent with later power-law scaling $\text{MSD} \propto \tau^\alpha$.

4. ✔ **Diffusive/superdiffusive scaling statements** (Sec. 1, p.2; Sec. 3.1, p.5)
   - **Claim:** States classical diffusion gives $\text{MSD} \propto \tau$, ballistic gives $\text{MSD} \propto \tau^2$, and superdiffusion corresponds to $\alpha > 1$ in $\text{MSD} \propto \tau^\alpha$.
   - **Checks:** limiting/sanity check
   - **Verdict:** PASS; confidence: high; impact: moderate
   - **Assumptions/inputs:** Standard interpretation of α in MSD scaling
   - **Notes:** These regimes are consistent with the MSD definition and with $\alpha_{\text{total}} \approx 1.82$ being between diffusive and ballistic.

5. ✔ **VACF normalization** (Eq. (3), Sec. 2.3, p.4)
   - **Claim:** Defines $C_v(\tau) = \langle\mathbf{v}(t) \cdot \mathbf{v}(t+\tau)\rangle / \langle|\mathbf{v}(t)|^2\rangle$.
   - **Checks:** dimensional consistency, normalization check
   - **Verdict:** PASS; confidence: high; impact: moderate
   - **Assumptions/inputs:** $\mathbf{v}(t)$ is a velocity estimate derived from $\mathbf{r}(t)$ using a consistent finite-difference scheme
   - **Notes:** C_v is dimensionless; if the averaging is consistent, $C_v(0) = 1$ follows.

6. ✔ **Step displacement definition** (Sec. 2.3, p.4)

- **Claim:** Defines step vectors $\Delta\mathbf{r}i = \mathbf{r}_i$.} - \mathbf{r
- **Checks:** definition consistency
- **Verdict:** PASS; confidence: medium; impact: minor
- **Assumptions/inputs:** Index i refers to consecutive snapshots along a trajectory, Periodic minimum-image convention is applied consistently when differencing positions
- **Notes:** Definition is standard; correctness depends on consistent periodic handling, which is stated for tracking but not explicitly for differencing.

7. ✖ **Gaussian fit applied to step magnitudes** (Sec. 2.3, p.4; Sec. 3.2, p.5)

   - **Claim:** Compares the distribution of $|\mathbf{\Delta r}|$ to a Gaussian and uses a K–S test to claim it is not distinguishable from Gaussian, ruling out Lévy flights.
   - **Checks:** distribution/normalization sanity check, logical consistency of inference
   - **Verdict:** FAIL; confidence: high; impact: critical
   - **Assumptions/inputs:** $|\mathbf{\Delta r}|$ denotes Euclidean norm of a 3D displacement vector
   - **Notes:** A Gaussian model is not a mathematically appropriate reference distribution for a nonnegative magnitude $|\mathbf{\Delta r}|$ derived from (possibly) Gaussian vector components; thus the stated hypothesis test is mismatched to the variable and the inference about Lévy flights is not supported as written.

8. ⚠ **Kurtosis statement for step-size distribution** (Sec. 3.2, p.5)

   - **Claim:** States the step-size distribution is nearly Gaussian with kurtosis < 0.5.
   - **Checks:** definition clarity, consistency with stated tested variable
   - **Verdict:** UNCERTAIN; confidence: medium; impact: moderate
   - **Assumptions/inputs:** Kurtosis definition (raw vs excess) is not specified, Variable for kurtosis ($|\mathbf{\Delta r}|$ vs components) is not specified
   - **Notes:** If kurtosis refers to excess kurtosis of step components, '< 0.5' is interpretable; if it refers to $|\mathbf{\Delta r}|$, the reference value for "Gaussian-like" differs. The paper does not specify which.

9. ⚠ **Vorticity-weighted centroid definition missing** (Sec. 2.2, p.3)

   - **Claim:** Defines vortex position by a vorticity-weighted centroid $\mathbf{r}_c$.
   - **Checks:** definition completeness, symbol/notation consistency
   - **Verdict:** UNCERTAIN; confidence: high; impact: moderate
   - **Assumptions/inputs:** A weighting function over the identified connected region exists (e.g., $|\omega|$, ω^2, enstrophy) but is not stated

- **Notes:** Without an explicit formula, $\mathbf{r}_c$ is not uniquely defined (magnitude vs signed vorticity; handling of multi-component ω; normalization). This affects what $\mathbf{r}(t)$ mathematically represents in Eqs. (2)–(3).

10. ⚠ **Decomposition into parallel/perpendicular motion relative to ω** (Sec. 2.4, p.4; Sec. 3.3, p.6)
    - **Claim:** Decomposes $\Delta\mathbf{r}$ into $\Delta\mathbf{r}\parallelland\Delta\mathbf{r}\perprelativetolocalvorticityvector\omega$ and compares mean magnitudes.
    - **Checks:** definition completeness, dimensional/geometry sanity check
    - **Verdict:** UNCERTAIN; confidence: medium; impact: minor
    - **Assumptions/inputs:** Whether ω is normalized to $\hat{\omega}$ is not stated, Handling of cases with $|\omega| = 0$ is not stated
    - **Notes:** Geometrically, the decomposition requires a unit direction $\hat{\omega}$; otherwise $\Delta\mathbf{r}\parallel asa“component”isambiguous.Thepaperreports|\Delta\mathbf{r}\parallel| and|\Delta\mathbf{r}_\perp|$ but does not give the projection formulas.

11. ✔ **Vorticity definition** (Sec. 2.4, p.4)
    - **Claim:** Defines $\omega = \nabla \times \mathbf{v}$.
    - **Checks:** definition consistency, dimensional consistency
    - **Verdict:** PASS; confidence: high; impact: minor
    - **Assumptions/inputs:** Consistent coordinate conventions for curl
    - **Notes:** Standard definition; units are 1/time given $\mathbf{v}$ has length/time.

12. ⚠ **Residual-trajectory MSD after subtracting large-scale advection** (Sec. 3.4, p.7)
    - **Claim:** Recomputes MSD after subtracting a large-scale advective velocity field and reports $\alpha_{\text{residual}} \approx \mathbf{1.93}$.
    - **Checks:** definition completeness, algebraic consistency with MSD definition
    - **Verdict:** UNCERTAIN; confidence: medium; impact: moderate
    - **Assumptions/inputs:** A well-defined large-scale velocity field U_L exists, Residual positions/steps are formed consistently with Eq. (2)
    - **Notes:** The filtering/subtraction operator is not defined; therefore it is not possible to verify that the residual MSD corresponds to the same mathematical object as the original MSD (e.g., subtracting velocities vs subtracting displacements yields different residual trajectories).

Limitations

- Only the parsed text provided (9 pages) was available; figures, any appendices, and any equations not present in the text could not be audited.

- Several method-critical quantities (vorticity-weighted centroid formula, ω-parallel/perpendicular decomposition formulas, definition of large-scale advective field) are referenced but not explicitly defined, preventing full symbolic verification of related claims.
- No numerical values, fits, or statistical test outputs were checked, per scope; only analytic validity and definition consistency were assessed.

Numerical results audit

This section audits **numerical/empirical** consistency: reported metrics, experimental design, baseline comparisons, statistical evidence, leakage risks, and reproducibility.

8 candidate checks were executed: 6 PASS, 1 FAIL, and 1 UNCERTAIN. The main numerical mismatch is a rounding/summary inconsistency for the reported diffusive exponent between Results and Conclusions; other audited items (range checks and one ratio recalculation) were consistent within stated tolerances.

Checked items

1. ✔ **C1** (Page 3 (Methods 2.1))
   - **Claim:** Physical spacing between grid points is $dx = 0.0078125$.
   - **Checks:** exact_fraction_check
   - **Verdict:** PASS
   - **Notes:** dx matched $1/128$ exactly (fraction and float exact match).
2. ⚠ **C2** (Page 3 (Methods 2.2))
   - **Claim:** Q-criterion thresholds use $k \in 2.5, 3.0, 4.0$.
   - **Checks:** set_membership_consistency
   - **Verdict:** UNCERTAIN
   - **Notes:** Cross-occurrence consistency cannot be evaluated because only one occurrence was provided in the inputs.
3. ✔ **C3** (Page 6 (Results 3.3, Table 1))
   - **Claim:** Table 1 lists p-values: v_x $p = 4 \times 10^{-17}$; v_y $p \approx 0$; v_z $p = 4 \times 10^{-4}$; Speed $p \approx 0$.
   - **Checks:** p_value_range_check
   - **Verdict:** PASS
   - **Notes:** All p-values satisfied $0 \leq p \leq 1$ under the bounds check (with ≈ 0 treated as 0 for bounds only).
4. ✔ **C4** (Page 6 (Results 3.3, Table 1))
   - **Claim:** Table 1 Pearson r values: v_x 0.139, v_y 0.790, v_z 0.059, Speed 0.336.
   - **Checks:** correlation_range_check
   - **Verdict:** PASS

- **Notes:** All correlations were within $[-1, 1]$.

5. ✔ **C5** (Page 6 (Results 3.3, anisotropy relative to vorticity axis))

   - **Claim:** Mean $|\Delta r_{\parallel}| = 0.001800$; Mean $|\Delta r_{\perp}| = 0.002131$; ratio (parallel/perpendicular) is 0.845.
   - **Checks:** ratio_recalculation
   - **Verdict:** PASS
   - **Notes:** Recomputed ratio 0.8446738620 agreed with reported 0.845 within tolerance.

6. ✖ **C6** (Page 5 (Results 3.1) and Page 7 (Conclusions))

   - **Claim:** Diffusive exponent reported as $\alpha_{\text{total}} = 1.815 \pm 0.009$ in Results; later summarized as $\alpha \approx 1.82$ in Conclusions.
   - **Checks:** rounded_value_consistency
   - **Verdict:** FAIL
   - **Notes:** The conclusion value was within the stated uncertainty band, but did not match the check's two-decimal rounding result for 1.815 (computed round(α_{total},2)=1.81), triggering failure under the specified rounding-consistency criterion.

7. ✔ **C7** (Page 5 (Results 3.1))

   - **Claim:** Directional exponents: $\alpha_x = 1.724 \pm 0.016$; $\alpha_y = 1.871 \pm 0.003$; $\alpha_z = 1.849 \pm 0.009$; text says transport most pronounced in y and least in x.
   - **Checks:** ordering_consistency
   - **Verdict:** PASS
   - **Notes:** Central values satisfied $\alpha_y > \alpha_z > \alpha_x$; y was max and x was min.

8. ✔ **C8** (Page 5 (Results 3.1))

   - **Claim:** Reported R^2 values: total $R^2 = 0.999$; x $R^2 = 0.996$; y $R^2 = 1.000$; z $R^2 = 0.999$.
   - **Checks:** r_squared_range_check
   - **Verdict:** PASS
   - **Notes:** All R^2 values satisfied $0 \leq R^2 \leq 1$.

Limitations

- Only parsed text was available; no access to underlying trajectory data, MSD/VACF arrays, or any supplementary tables needed to recompute fitted exponents, uncertainties, kurtosis, and statistical test results.
- No figure pixel/value extraction was performed (and is excluded by scope), so any numeric values that might appear only in plots cannot be audited.

- Several claims are qualitative (e.g., robustness across thresholds/trajectory lengths) without providing the corresponding numeric results, preventing fast internal consistency checks beyond simple parameter-set consistency.