Skepthical review: *Anomalous Transport and Ergodicity in Chaotic Point-Vortex Systems: A Comparison with Lévy Walks*

Summary

This manuscript numerically studies passive tracer transport in two-dimensional chaotic point-vortex flows and compares it to canonical Lévy-walk dynamics (Secs. 2–3). By varying the number of vortices N in a Hamiltonian point-vortex model and the Lévy-walk flight-time exponent β, the authors span regimes from near-normal diffusion to strong superdiffusion. They analyze ensemble MSD scaling, VACF, residence-time statistics based on a low-speed threshold, displacement PDFs (including Lévy-stable fits), and an ergodicity-breaking (EB) parameter derived from TAMSD variability (Eqs. (3)–(6), Secs. 3.2–3.7). The main kinematic message—that higher-N vortex flows show Lévy-like signatures (superdiffusive MSD, heavy-tailed PDFs, power-law residence times) similar to Lévy walks with $1 < \beta < 2$—is plausible and potentially interesting. The central conceptual claim is that matching kinematic diagnostics does not imply equivalence in ergodicity: EB decreases with N in the vortex system but increases with stronger Lévy-walk superdiffusion (Sec. 3.7). However, the quantitative robustness of the exponent estimates, tails, and especially the EB trends is currently difficult to assess due to limited sampling, incomplete model/numerics specification (domain, boundary handling, singularity treatment, integrator), and insufficiently rigorous uncertainty quantification and fitting methodology. Addressing these points would substantially strengthen the paper's main conclusion about the limits of Lévy-walk coarse-graining for deterministic chaotic advection (Sec. 4).

Strengths

- Clear comparative framing between a deterministic Hamiltonian transport model (point vortices) and a benchmark stochastic process (Lévy walks), with tunable controls N and β (Secs. 1–2).
- Broad and relevant diagnostic suite—MSD, VACF, displacement PDFs, residence-time statistics, TAMSD/EB—so the comparison is not based on a single metric (Sec. 2.3, Sec. 3).
- Compelling qualitative evidence that increasing N produces Lévy-like kinematic signatures (superdiffusive MSD, heavy-tailed displacement statistics, long residence times) (Secs. 3.2–3.5).
- Conceptually interesting observation that "superdiffusion strength" and "ergodicity breaking" can trend oppositely in deterministic chaotic advection versus Lévy-walk models, summarized in a regime map (Sec. 3.7).
- Core definitions for MSD/VACF/TAMSD/EB are largely standard and internally consistent (Eqs. (3)–(6)), making the intended computations reproducible once implementation details are added.

Major issues

1. **Sampling/uncertainty is insufficient for the paper's central quantitative claims (especially EB trends and tail exponents). The point-vortex results appear to use very small tracer ensembles (e.g., 5 trajectories per N) over relatively short durations (Sec. 2.1.1, Secs. 3.6–3.7). EB, power-law tails, and stable fits are high-variance and sensitive to rare events and finite-time effects, so it is currently unclear whether reported monotonic trends with N are statistically significant or could change with more realizations/longer T. It is also unclear whether "ensemble" averages mix multiple independent vortex realizations or only multiple tracers in a single vortex realization (flow-to-flow variability vs tracer-to-tracer variability).**

   Recommendation: Increase statistical power and report uncertainty throughout: (i) substantially increase tracer count per N and, importantly, include multiple independent vortex realizations per N; (ii) report confidence intervals (e.g., bootstrap over trajectories and over vortex realizations) for MSD exponents α, residence-time exponents γ, stable-fit parameters, and EB plateau values; (iii) add finite-time checks by repeating analyses for multiple observation times T (or subtrajectory analysis) to show convergence of $\alpha(T)$ and $\mathrm{EB(T)}$. Clearly state the number of independent vortex realizations and tracers used in each figure/table.

2. **Model specification is incomplete in ways that can materially affect transport scaling: domain size, boundary conditions, and handling of particles/vortices near or beyond the domain are not clearly specified (Sec. 2.1.1). This is critical for long-time displacement statistics and MSD scaling. In addition, units/nondimensionalization are ambiguous: the Lévy walk uses $v_0 = 1.0$ m/s with times in seconds (Sec. 2.2), while vortex circulations Γ_k are drawn from a standard normal distribution with no stated units/scale, making the dimensional consistency of Eq. (1) and all reported times/lengths unclear (Secs. 2.1.1–2.2).**

   Recommendation: Explicitly state: (i) domain geometry and size (e.g., $[0, L] \times [0, L]$ or centered square), (ii) boundary conditions (free-space, periodic, reflecting, etc.), and (iii) what happens when tracers/vortices approach boundaries or leave the domain. Provide a clear nondimensionalization (preferred) or a consistent physical scaling: define the length scale L, time scale, and circulation scale for Γ_k so that Eq. (1) matches the reported time/velocity units. Update figure axes/captions accordingly.

3. **Numerical integration and singularity treatment for point vortices are under-described. Point-vortex-induced velocities diverge as $1/r$; without careful handling, rare close approaches can dominate VACF, residence-time thresholding, displacement tails, and EB (Sec. 2.1.1, Eq. (1)). The**

manuscript does not specify the integrator, timestep control/accuracy checks, whether any vortex-core regularization/desingularization is used, or diagnostics confirming results are not artifacts of near-singular events.

Recommendation: Add a dedicated numerical-methods subsection for the vortex simulations: integrator type (e.g., RK4/symplectic), error control, timestep sensitivity tests, and treatment of singularities (e.g., vortex core radius a, velocity cap, or desingularized kernel). Report at least one robustness check: vary Δt and (if used) core size a and show α, EB, and tail statistics are stable; additionally report statistics of minimum tracer–vortex distance and/or the distribution of peak speeds to demonstrate tails are not dominated by numerical blow-ups.

4. **Power-law/tail fitting methodology is not sufficiently rigorous or reproducible, and key inferred relations are overstated. Residence times are fit as $P(\tau_{\text{res}}) \sim \tau_{\text{res}}^{-\gamma}$ (Sec. 3.4), but the fitting procedure (MLE vs linear regression), fit ranges, binning, and goodness-of-fit are not specified, nor are alternative heavy-tailed models (truncated power law, lognormal, stretched exponential) compared. Moreover, the paper asserts a specific scaling relation between γ and α ($\alpha = 2 - (\gamma - 1)$ in Sec. 3.4, p.9), even though τ_{res} is defined via a low-velocity thresholding rule rather than an explicitly justified renewal waiting-time variable; the noted mismatch between predicted and measured α is currently waved through as "reasonable agreement."**

   Recommendation: Make tail inference reproducible and appropriately qualified: (i) specify the estimator (preferably MLE-based with objective selection of lower cutoff), (ii) report fit ranges/cutoffs, uncertainty on γ, and goodness-of-fit and/or likelihood comparisons against plausible alternatives; (iii) either derive the α–γ relation under explicit assumptions linking τ_{res} to a renewal waiting-time (with citations), or reframe it as an empirical correlation and avoid presenting a specific formula as theoretically implied by the current τ_{res} definition.

5. **EB estimation/"plateau" definition is fragile given finite-time effects and small ensembles (Secs. 3.6–3.7, Eq. (6)). Defining a plateau as the mean EB over the final 20% of lag times can be dominated by noise and by the shrinking $(T - \Delta)$ sample size at large Δ. Without uncertainty bands and convergence checks, the headline EB-vs-N trend is difficult to trust.**

   Recommendation: Strengthen EB analysis: (i) plot EB versus normalized lag Δ/T and include uncertainty bands (bootstrap); (ii) test sensitivity of the plateau definition (final 10%/20%/30%, or fitting to an asymptotic form); (iii) perform a convergence study varying T (or using block/subtrajectory analysis) to show $\text{EB}(\Delta, \text{T})$ approaches a stable curve; (iv) for Lévy walks, validate the EB implementation against known theoretical or benchmark numerical results (where available) as an additional sanity check beyond MSD.

6. **Displacement PDF interpretation and Lévy-stable fitting need more careful justification, especially for Lévy walks. Lévy-walk propagators generally differ from α-stable Lévy-flight distributions due to finite speed and space–time coupling; ballistic fronts/peaks can occur and the "stable" fit may only apply in limited regimes. For the vortex system, claiming "approximately Lévy-stable tails" (Sec. 3.5) is currently based on limited fitting detail and mostly visual agreement.**

   Recommendation: Clarify what is being fit and where: (i) specify the fitting method, the Δx range used (core vs tails), and uncertainties on stable parameters; (ii) explicitly discuss expected Lévy-walk propagator features (finite-speed effects, possible fronts) and whether/why an α-stable fit is appropriate for the chosen times/lags; (iii) add scaling-collapse tests (e.g., plotting $P(\Delta x, t)$ under the expected rescaling) and/or directly estimate tail exponents without assuming a stable family; (iv) for cross-model comparison, consider quantitative mismatch metrics (e.g., Wasserstein/KL on increment PDFs) after calibrating β to match α.

7. **Figure/documentation issues materially reduce interpretability and reproducibility. Several key figures (notably Figs. 1–6) lack explicit axis units/normalizations, domain extents, consistent labeling across panels, and uncertainty visualization; at least one figure is reported as potentially rendering blank (Fig. 4). Captions sometimes state theoretical expectations or normalizations that are not clearly verifiable from the plot content (Secs. 3.2–3.5).**

   Recommendation: Revise Figs. 1–6 for actionability: add axis units (or nondimensional units), define any normalization in captions, state domain/boundary conditions where relevant, harmonize axis limits/aspect ratios for cross-panel comparison, and add uncertainty bands/error bars for fitted quantities. Ensure all figures render correctly in the submission pipeline (vector PDF preferred) and that captions precisely match what is plotted (including which data subsets and fit ranges are used).

Minor issues

1. The operational definition of "trapping" via low speed (below the 25th percentile of pooled speeds per N) is not clearly linked to a dynamical trapping mechanism in Hamiltonian point-vortex flows (Sec. 3.4). Low speed can reflect many situations (e.g., far-field cancellation) and may not uniquely correspond to sticking near coherent structures; results may depend strongly on the chosen quantile and on pooling choices.

   Recommendation: Test robustness of residence-time results to the threshold definition: repeat key plots for multiple quantiles (e.g., 10%, 20%, 30%) and/or consider complementary, more dynamical criteria (e.g., proximity to vortices, curvature/turning-rate segmentation, or a coherent-structure indicator). Clearly state whether the percentile is computed per-trajectory, per-realization, or pooled across tracers, and discuss implications.

2. VACF interpretation includes potentially misleading wording: describing Lévy walks as "memoryless motion" conflicts with their persistent-flight velocity correlations (Sec. 3.3). Also, the combination "faster decorrelation with larger N yet stronger superdiffusion" is nontrivial and would benefit from a clearer intermittency/coherent-flight interpretation rather than decorrelation-time intuition alone.

   Recommendation: Rephrase Lévy-walk dynamics as renewal at turning events (not memoryless velocity), and add a short interpretive paragraph connecting vortex VACF behavior to intermittency/heavy-tailed increments (e.g., rare long flights) as a mechanism for superdiffusion even when short-time correlations decay faster.

3. Cross-model matching is partly qualitative: at times the manuscript implies an analogy between N and β via matching α, but then compares other diagnostics without a clear calibration strategy (Secs. 3.2–3.7).

   Recommendation: Adopt an explicit calibration protocol (e.g., choose β to match measured α for each N) and then compare other statistics "out of sample," ideally adding one quantitative mismatch metric (distance between increment PDFs, VACF discrepancy) to support the regime-map narrative.

4. Some reporting gaps and internal-document consistency points hinder clarity: provide exact Lévy-walk ensemble sizes and simulation lengths (Sec. 2.2); clarify use of overlapping windows for MSD/TAMSD and any bias from correlations (Sec. 2.3); Table 2 reportedly omits $N = 5$ while other sections discuss $N = 5$.

   Recommendation: Add a compact simulation-parameter table covering both models (N, number of vortex realizations, number of tracers, T, Δt, domain/boundaries; and for Lévy walks: number of walkers, β values, T, Δt, and any truncations). Ensure tables consistently include all N values analyzed or explicitly justify omissions.

5. Presentation/formatting: the author/affiliation line as described in the unstructured report appears nonstandard for scholarly publication.

   Recommendation: Replace with standard author and affiliation formatting consistent with the target journal/conference guidelines.

Very minor issues

1. Notation ambiguity: α is used for MSD scaling, while α_{stable} is used for Lévy-stable stability index (Sec. 3.5, p.9), which can be misread as the same quantity.

   Recommendation: Use distinct symbols (e.g., α_{MSD} vs μ_{stable}) or add an explicit reminder near the first stable-fit discussion that the parameters are unrelated.

2. EB definition typesetting could be misread without explicit parentheses (Eq. (6)).

   Recommendation: Typeset Eq. (6) with explicit parentheses: $\mathrm{EB} = (\langle(\delta^2)^2\rangle - \langle\delta^2\rangle^2)/\langle\delta^2\rangle^2$, and define clearly what averages are taken over (trajectory ensemble, time origins, vortex realizations).

3. Figure accessibility/legibility: small fonts, dense multi-panels, and reliance on color alone reduce readability (Figs. 1–6). Missing subpanel lettering and occasional low-resolution exports make cross-referencing harder.

   Recommendation: Increase font/line sizes; use colorblind-safe palettes with redundant encodings (line styles/markers); add consistent subpanel labels (a,b,c,...) and on-panel parameter annotations; export figures as vector graphics where possible.

4. Minor stylistic/consistency issues (terminology, redundant labels, tick formatting, legends overlapping data) reduce polish but do not change conclusions.

   Recommendation: Standardize terminology (e.g., VACF notation), simplify legends, ensure axis scaling (log/linear) is explicitly indicated, and avoid overlapping legend boxes.

Key statements and references

- • **For the standard Lévy walk model with constant speed v_0 and flight times τ drawn from a heavy-tailed distribution $P(\tau) \sim \tau^{-\beta-1}$ for $\tau \geq \tau_{\min}$, the transport regime is determined by β such that for $1 < \beta < 2$ the process is superdiffusive with theoretical mean-squared-displacement exponent $\alpha = 3 - \beta$, whereas for $\beta > 2$ the mean flight time is finite and the process becomes normally diffusive with $\alpha = 1$.**
- *Reference(s):* 2
- • **According to continuous-time random walk theory, a power-law residence-time distribution $P(\tau_{\mathrm{res}}) \sim \tau_{\mathrm{res}}^{-\gamma}$ with $1 < \gamma < 2$ implies a divergent mean residence time and leads to superdiffusive transport with anomalous diffusion exponent related by $\alpha = 2 - (\gamma - 1)$, whereas $\gamma > 2$ corresponds to a finite mean trapping time and near-normal diffusion.**
- *Reference(s):* 4
- • **For Lévy walks, stronger superdiffusion (smaller tail exponent β) is theoretically associated with stronger ergodicity breaking, such that the ergodicity-breaking parameter EB increases as β decreases from the normal-diffusive regime ($\beta > 2$) to strongly superdiffusive cases ($1 < \beta < 2$), because the dynamics become dominated by rare, extremely long ballistic flights that enhance trajectory-to-trajectory variability.**
- *Reference(s):* 6

Mathematical consistency audit

This section audits **symbolic/analytic** mathematical consistency (algebra, derivations, dimensional/unit checks, definition consistency).

Maths relevance: light

The paper contains a small set of standard transport-statistics definitions (MSD, VACF, TAMSD, EB) and a few scaling-law claims relating power-law exponents to anomalous diffusion. The main analytic concern is an asserted scaling relation linking a residence-time tail exponent to the MSD exponent without specifying the underlying theoretical assumptions needed to make that link verifiable.

Checked items

1. ✔ **Point-vortex tracer advection equation** (Eq. (1), Sec. 2.1.1, p.3)

    - **Claim:** Tracer velocity is the superposition of point-vortex induced velocities: $\dot{\mathbf{r}} = \sum \Gamma_k/(2\pi)\hat{z} \times (\mathbf{r} - \mathbf{r}_k)/|\mathbf{r} - \mathbf{r}_k|^2$.
    - **Checks:** algebra/structure, dimensional/units consistency, notation consistency
    - **Verdict:** PASS; confidence: high; impact: moderate
    - **Assumptions/inputs:** Circulation Γ_k has the usual physical dimension of area/time if $\mathbf{r}$ is a length., The 2D cross product with $\hat{z}$ is interpreted as a 90° rotation in the plane.
    - **Notes:** Vector form and scaling with $1/|\mathbf{r} - \mathbf{r}_k|$ are consistent; dimensional check passes if Γ_k carries appropriate units. The paper later mixes physical units with Γ_k drawn from $N(0, 1)$ without stating nondimensionalization (flagged separately).

2. ✔ **Lévy-walk flight-time heavy-tail definition** (Eq. (2), Sec. 2.1.2, p.3)

    - **Claim:** Flight times satisfy $P(\tau) \sim \tau^{-\beta-1}$ for $\tau \geq \tau_{\min}$.
    - **Checks:** definition consistency, normalization/constraints
    - **Verdict:** PASS; confidence: medium; impact: minor
    - **Assumptions/inputs:** $P(\tau)$ is a probability density; '$\sim$' denotes asymptotic tail behavior.
    - **Notes:** Asymptotic form is consistent as a tail statement. Normalization requires $\beta > 0$, which is satisfied by the β values used in the paper.

3. ✔ **Stated Lévy-walk MSD scaling exponent relation** (Sec. 2.1.2, p.3 (text following Eq. (2)); reiterated Sec. 3.2, p.7)

    - **Claim:** For $1 < \beta < 2$, the Lévy walk yields superdiffusion with $\alpha = 3 - \beta$; for $\beta > 2$, $\alpha = 1$.
    - **Checks:** internal consistency, limiting/sanity cases
    - **Verdict:** PASS; confidence: low; impact: moderate

- **Assumptions/inputs:** Constant-speed flights with direction resets; α denotes MSD exponent via $\langle \Delta r^2 \rangle \propto t^\alpha$.
- **Notes:** No derivation is provided in the paper, so only internal sanity checks are possible: $\beta \to 2$ gives $\alpha \to 1$ and $\beta \to 1$ gives $\alpha \to 2$ (ballistic), which are consistent with the regime descriptions.

4. ✔ **Ensemble-averaged MSD definition** (Eq. (3), Sec. 2.3.1, p.4)

- **Claim:** $\langle \Delta r^2(\Delta t) \rangle = \langle |\mathbf{r}(t + \Delta t) - \mathbf{r}(t)|^2 \rangle$ averaged over start times and trajectories.
- **Checks:** definition consistency
- **Verdict:** PASS; confidence: high; impact: minor
- **Assumptions/inputs:** Time lag Δt is within observation window.
- **Notes:** Definition is standard and internally consistent with later use of α as the log-log slope.

5. ✔ **Normalized velocity autocorrelation definition** (Eq. (4), Sec. 2.3.2, p.4)

- **Claim:** $C_v(\Delta t) = \langle \mathbf{v}(t) \cdot \mathbf{v}(t + \Delta t) \rangle / \langle |\mathbf{v}(t)|^2 \rangle$.
- **Checks:** definition consistency, normalization/constraints
- **Verdict:** PASS; confidence: medium; impact: minor
- **Assumptions/inputs:** Averaging procedure over t/trajectories is consistent in numerator and denominator.
- **Notes:** Produces $C_v(0) = 1$ if the same averaging is used; consistent with a normalized VACF.

6. ✔ **Residence-time tail model and mean-divergence criterion** (Sec. 2.3.3 (definition), p.4; Sec. 3.4 (tail claim), p.9)

- **Claim:** Residence-time tails follow $P(\tau_{\text{res}}) \sim \tau_{\text{res}}^{-\gamma}$; for $1 < \gamma < 2$ the mean residence time diverges.
- **Checks:** normalization/constraints, limiting/sanity cases
- **Verdict:** PASS; confidence: high; impact: minor
- **Assumptions/inputs:** $P(\tau_{\text{res}})$ is a PDF with tail exponent γ., Tail behavior dominates large-τ integrals.
- **Notes:** For a tail $P(\tau) \sim \tau^{-\gamma}$, the mean behaves like $\int \tau \cdot \tau^{-\gamma} d\tau = \int \tau^{1-\gamma} d\tau$, which diverges for $\gamma \leq 2$. Thus the stated divergence for $1 < \gamma < 2$ is correct.

7. ⚠ **Claimed α–γ relation connecting trapping to MSD scaling** (Sec. 3.4, p.9 (text: 'This result can be connected... via the relation $\alpha = 2 - (\gamma - 1)$ for $1 < \gamma < 2$.'))

- **Claim:** If $1 < \gamma < 2$, then the anomalous diffusion exponent satisfies $\alpha = 2 - (\gamma - 1)$ (i.e., $\alpha = 3 - \gamma$).

- **Checks:** derivation logic, definition consistency
- **Verdict:** UNCERTAIN; confidence: medium; impact: moderate
- **Assumptions/inputs:** An implicit CTRW-like theory is applicable., The measured τ_{res} is the relevant renewal-time variable in that theory.
- **Notes:** The paper does not specify the stochastic model mapping from the threshold-defined τ_{res} (low-speed residence) to a renewal waiting-time or flight-time variable, nor does it provide a derivation of α as a function of γ. Without these missing steps/assumptions, the formula cannot be verified from the document alone.

8. ✔ **Time-averaged MSD (TAMSD) definition** (Eq. (5), Sec. 2.3.5, p.5)

   - **Claim:** For each trajectory i, $\delta_i^2(\Delta t) = (1/(T - \Delta t)) \int_0^{T-\Delta t} |\mathbf{r}_i(t' + \Delta t) - \mathbf{r}_i(t')|^2 dt'$.
   - **Checks:** definition consistency, normalization/constraints
   - **Verdict:** PASS; confidence: high; impact: minor
   - **Assumptions/inputs:** $0 < \Delta t < T$.
   - **Notes:** Integral bounds and normalization by $(T - \Delta t)$ are consistent with standard TAMSD definitions.

9. ✔ **EB parameter definition and equivalence of forms** (Eq. (6), Sec. 2.3.5, p.5)

   - **Claim:** $\text{EB}(\Delta t) = \text{Var}[\delta^2(\Delta t)]/(\text{Mean}[\delta^2(\Delta t)])^2 = (\langle(\delta^2)^2\rangle - \langle\delta^2\rangle^2)/\langle\delta^2\rangle^2$.
   - **Checks:** algebra between shown steps, definition consistency
   - **Verdict:** PASS; confidence: high; impact: minor
   - **Assumptions/inputs:** Mean and variance are taken over the ensemble of trajectories (index i).
   - **Notes:** Identity $\text{Var}[X] = \langle X^2\rangle - \langle X\rangle^2$ justifies the second form. Typesetting could be clarified with parentheses to avoid ambiguity.

10. ✔ **Statement $\text{EB} = 0$ indicates ergodicity (interpretation of EB)** (Sec. 2.3.5, p.5)

    - **Claim:** $\text{EB} = 0$ signifies perfect ergodicity; sustained $\text{EB} > 0$ indicates heterogeneity across trajectories.
    - **Checks:** definition/interpretation consistency
    - **Verdict:** PASS; confidence: medium; impact: minor
    - **Assumptions/inputs:** Ergodicity is operationalized as equivalence of time-averaged and ensemble-averaged behavior across realizations, measured via TAMSD variability.
    - **Notes:** Given EB is a normalized variance across trajectory-wise TAMSDs, $\text{EB} = 0$ indeed means all trajectories share identical TAMSD at that lag, consistent with the stated operational interpretation.

11. ⚠ **Use of Lévy-stable fits for displacement PDFs vs finite-speed constraint** (Sec. 2.3.4, p.5 and Sec. 3.5, p.9)

   - **Claim:** Long-lag displacement PDFs are compared to Gaussian and symmetric Lévy-stable distributions; the $\beta = 1.5$ Lévy walk displacement PDF is described by a Lévy-stable distribution with index $\alpha_{\text{stable}} \approx 1.5$.
   - **Checks:** constraint/sanity checks, notation consistency
   - **Verdict:** UNCERTAIN; confidence: low; impact: minor
   - **Assumptions/inputs:** Comparison is meant as an empirical fit/approximation over the sampled range.
   - **Notes:** A constant-speed walk over a finite lag Δt implies a hard bound $|\Delta x| \leq v_0 \Delta t$, while an ideal Lévy-stable law has unbounded support. If the authors mean approximate core/tail-shape fitting on a finite interval, this is fine, but the paper does not clarify this analytic distinction.

12. ⚠ **Units/nondimensionalization consistency for vortex vs Lévy-walk datasets** (Secs. 2.1.1–2.2, pp.3–4)

   - **Claim:** Vortex simulations and Lévy walks are both discussed with times in seconds, and Lévy walks with v_0 in m/s, while Γ_k is sampled from a standard normal distribution.
   - **Checks:** dimensional/units consistency, definition consistency
   - **Verdict:** UNCERTAIN; confidence: medium; impact: moderate
   - **Assumptions/inputs:** Either all variables are nondimensional, or Γ_k has physical units consistent with Eq. (1).
   - **Notes:** Eq. (1) is dimensionally consistent only if Γ_k carries appropriate units and $\mathbf{r}$ is in consistent length units. The paper does not state any nondimensionalization or physical scaling for Γ_k and the domain, making the use of 'm/s' and 's' potentially inconsistent.

Limitations

- Only the mathematics and definitions explicitly present in the provided PDF text were checked; no external theory or derivations were imported.
- Several scaling-law statements (e.g., $\alpha = 3 - \beta$ and the asserted α–γ relation) are not derived in the paper; such items can only be assessed for internal consistency and basic sanity limits, otherwise marked UNCERTAIN.
- Figure-based empirical fit claims (e.g., Lévy-stable fit quality) cannot be validated analytically beyond checking basic constraints implied by the model definitions (e.g., finite-speed bounds).

Numerical results audit

This section audits **numerical/empirical** consistency: reported metrics, experimental design, baseline comparisons, statistical evidence, leakage risks, and reproducibility.

Twelve numeric/logical consistency checks derived from the text and tables were executed; all 12 passed with no detected discrepancies under the stated tolerances. Several additional quantitative claims remain unverified because they require extracting values from figures or fitting to underlying time-series/PDF data not available in the parsed text.

Checked items

1. ✔ **C01__time__steps__total__samples** (Page 3, Section 2.2 (Simulation datasets))
    - **Claim:** Each trajectory was integrated for a total duration of 24.75 s with a time step of 0.05 s.
    - **Checks:** duration__step__count__consistency
    - **Verdict:** PASS
    - **Notes:** $T/\Delta t = 24.75/0.05 = 495$ steps exactly; implies 496 stored points if including endpoints.

2. ✔ **C02__total__tracer__trajectories__count** (Page 3, Section 2.2 (Simulation datasets))
    - **Claim:** We simulated the trajectories of 5 passive tracers for each of the four vortex configurations ($N = 5, 10, 20, 40$).
    - **Checks:** count__product__total
    - **Verdict:** PASS
    - **Notes:** Computed total trajectories = $5 \times 4 = 20$.

3. ✔ **C03__levy__alpha__from__beta__1p2** (Page 3, Section 2.1.2 (Lévy walk model) and Page 4, Section 2.2 (β list))
    - **Claim:** For $1 < \beta < 2$, the model produces superdiffusion with theoretical mean squared displacement exponent $\alpha = 3 - \beta$. Dataset includes $\beta = 1.2$.
    - **Checks:** formula__substitution
    - **Verdict:** PASS
    - **Notes:** $\alpha = 3 - 1.2 = 1.8$.

4. ✔ **C04__levy__alpha__from__beta__1p5** (Page 3, Section 2.1.2 and Page 4, Section 2.2)
    - **Claim:** For $1 < \beta < 2$, $\alpha = 3 - \beta$. Dataset includes $\beta = 1.5$.
    - **Checks:** formula__substitution
    - **Verdict:** PASS
    - **Notes:** $\alpha = 3 - 1.5 = 1.5$.

5. ✔ **C05__levy__alpha__from__beta__1p8** (Page 3, Section 2.1.2 and Page 4, Section 2.2)
    - **Claim:** For $1 < \beta < 2$, $\alpha = 3 - \beta$. Dataset includes $\beta = 1.8$.
    - **Checks:** formula__substitution

- **Verdict:** PASS
- **Notes:** $\alpha = 3 - 1.8 = 1.2$.

6. ✔ **C06_levy_beta_gt2_implies_alpha1_for_2p5** (Page 3, Section 2.1.2 and Page 4, Section 2.2)

    - **Claim:** For $\beta > 2$, normal diffusion with $\alpha = 1$. Dataset includes $\beta = 2.5$ and text notes normal diffusion ($\alpha = 1.0$).
    - **Checks:** conditional_rule_check
    - **Verdict:** PASS
    - **Notes:** $\beta = 2.5$ satisfies $\beta > 2$ and the stated $\alpha = 1.0$ matches the rule.

7. ✔ **C07_beta_to_alpha_span_claim** (Page 4, Section 2.2 (Lévy walk dataset description))

    - **Claim:** $\beta = 1.2, 1.5, 1.8, 2.5$ were chosen to span strong superdiffusion ($\alpha = 1.8$) to normal diffusion ($\alpha = 1.0$).
    - **Checks:** derived_range_endpoints
    - **Verdict:** PASS
    - **Notes:** Endpoints consistent with stated theory: $\beta = 1.2 \to \alpha = 1.8$; $\beta = 2.5 \to \alpha = 1.0$.

8. ✔ **C08_msd_trapping_relation_alpha_from_gamma_N40** (Page 9, Section 3.4 (Lévy-like trapping statistics))

    - **Claim:** Relation given: $\alpha = 2 - (\gamma - 1)$ for $1 < \gamma < 2$. For $N = 40$, $\gamma \approx 1.7$ implies $\alpha \approx 2 - (1.7 - 1) = 1.3$.
    - **Checks:** formula_substitution
    - **Verdict:** PASS
    - **Notes:** Worked arithmetic is consistent: $\alpha = 2 - (1.7 - 1) = 1.3$.

9. ✔ **C09_gamma_uncertainty_range_condition_N40** (Page 9, Table 2 and Section 3.4)

    - **Claim:** For $N = 40$, $\gamma = 1.7 \pm 0.2$ and claim emphasizes $\gamma < 2$ (divergent mean residence time).
    - **Checks:** inequality_with_uncertainty
    - **Verdict:** PASS
    - **Notes:** Conservative upper bound $1.7 + 0.2 = 1.9$ remains < 2.

10. ✔ **C10_gamma_uncertainty_range_condition_N10** (Page 9, Table 2 and Section 3.4)

    - **Claim:** For $N = 10$, $\gamma = 2.8 \pm 0.4$ and statement says $\gamma > 2$ (finite mean trapping time / exponential cutoff).
    - **Checks:** inequality_with_uncertainty

- **Verdict:** PASS
- **Notes:** Conservative lower bound $2.8 - 0.4 = 2.4$ remains > 2.

11. ✔ **C11__table1__monotonic__increase__alpha__with__N** (Page 7, Table 1)
    - **Claim:** Table 1 implies α increases monotonically with N: $(N, \alpha) = (5, 1.05), (10, 1.18), (20, 1.35), (40, 1.52)$.
    - **Checks:** monotonicity_check
    - **Verdict:** PASS
    - **Notes:** Verified strict monotonic increase: $1.05 < 1.18 < 1.35 < 1.52$.

12. ✔ **C12__table1__R2__range__check** (Page 7, Table 1)
    - **Claim:** Table 1 lists R^2 values: $0.97, 0.98, 0.97, 0.96$.
    - **Checks:** boundedness_check
    - **Verdict:** PASS
    - **Notes:** All listed R^2 values lie within $[0, 1]$.

Limitations

- Only parsed text (and not underlying numeric datasets) is available; many quantitative claims referenced to figures cannot be recomputed without the raw time series or tabulated fit results.
- Figure-based approximate values (e.g., VACF zero-crossings, EB plateaus, slope fits) cannot be verified without extracting curve data, which is out of scope.
- No checks were proposed that require external datasets, internet access, or long simulations; such items are instead listed as unverified.